Journal of Geometry and Physics 34 (2000) 206–225

On the quantum Lorentz group

M. Lagraa*

Laboratoire de Physique Théorique, Université d'Oran Es-Sénia, 31100 Algérie, Algeria

Received 14 April 1999; received in revised form 14 September 1999

Abstract

The quantum analog of Pauli matrices are introduced and investigated. From these matrices and an appropriate trace over spinorial indices we construct a quantum Minkowski metric. In this framework we show explicitly the correspondence between the $SL(2, C)$ and Lorentz quantum groups. Five $\mathcal{R}$ matrices of the quantum Lorentz group are constructed in terms of the R matrix of $SL(2, C)$ group. These $\mathcal{R}$ matrices satisfy Yang–Baxter equations and two of which have adequate properties tied to the quantum Minkowski space structure as the reality conditions of the coordinates and the symmetrization of the metric. It is also shown that the Minkowski metric leads to invariant and central lengths of four-vectors.

JGPSC: Quantum groups

MSC: 81R50; 16W30

Keywords: Quantum Lorentz group; Quantum Minkowski space

1. Introduction

The Lorentz group plays a fundamental role in physics. First, it constitutes the homogeneous part of Poincaré group which is intrinsically connected to the geometry of the space–time and leaves invariant all physical systems discribed by the special theory of relativity. Second, the different representations of Lorentz group are field discribing particles which constitute the physical systems. For these reasons, it is especially interesting to study the noncommutative version of the Lorentz group.

The other reason which makes the study of the quantum Lorentz group interesting is the fact that in quantum field theory based on a classical space–time and a classical Lorentz group there exist difficulties tied to small space–time distances. One may

* Laboratoire de Physique Théorique, Unité Mixte de Recherche du CNRS, UMR 8627, Batiment 210, Université Paris XI, 91405 Orsay, France.

0393-0440/00/$ – see front matter
PII: S0393-0440(99)00064-9

hope to solve these difficulties by new tools provided by the noncommutative geometry [1–4,15–17].

The noncommutative Minkowski space–time structure emerges from quantum Lorentz group which has been studied by many authors, either in the formalism of bimodules over $C^\star$ algebra [5,6] or from a purely “algebraic” point of view [7–10,18]. The latter way gives two Lorentz $\mathcal{R}$ matrices denoted $\mathcal{R}_{\rm I}$ and $\mathcal{R}_{\rm II}$ in [9].

In this paper one constructs quantum Lorentz group out of the $SL(2, C)$ group by showing how all properties of the former can be deduced from the latter which are well known. In particular we construct out of the R matrix of the quantum $SL(2, C)$ group five $\mathcal{R}$ matrices of the quantum Lorentz group which satisfy the Yang–Baxter equations. We show that only two of these $\mathcal{R}$ matrices have adequate properties tied to the quantum Minkowski space structure, namely, the symmetrization of the metric and the reality of the vectors constructed as bilinear of a spinor and a conjugate spinor.

The paper is organized in the following way. In Section 2 we recall the well known results provided by the bicovariant calculus over $SL(2, C)$ and $SU(2)$ quantum groups. We shall assume that the undotted (conjugate) and dotted generators of quantum $SL(2, C)$ group satisfy the commutation rules of the quantum $SU(2)$ group.

In Section 3 the construction of quantum Lorentz group is carried out of the quantum $SL(2, C)$ group following the analog of the homomorphism for the classical groups $SO(1, 3) \sim SL(2, C)\backslash Z_2$. We shall start by investigating the quantum analog of the Pauli matrices from which we construct the Minkowski metric and establish the completeness relation. From the properties of the quantum Pauli matrices and the generators of $SL(2, C)$, we contruct the generators of the quantum Lorentz group. We show that they satisfy the axiomatic structure of the Hopf algebras and the orthogonality relations. We also construct the $\mathcal{R}$ matrices of the Lorentz group out of those of $SL(2, C)$ group. These $\mathcal{R}$ matrices, denoted $\mathcal{R}^{\pm}_{({\rm I})}$, satisfy the Yang–Baxter equations, the Hecke relations and exhibit the quantum symetrization properties of the Minkowski metric.

In Section 4 we investigate the properties of the Minkowski space. In particular, we show that the Minowski metric induces an invariant length which commutes with the Hopf algebra $\mathcal{A}$ generated by the quantum $SL(2, C)$ group generators, the undotted and dotted basis (spinors) of the bicovariant bimodule over $\mathcal{A}$ and the quantum coordinates of the Minkowski space.

Finally, in addition to $\mathcal{R}^{\pm}_{({\rm I})}$, we construct and discuss in Section 5 other $\mathcal{R}$ matrices of the Lorentz group denoted $\mathcal{R}_{({\rm III})}$ and $\mathcal{R}^{\pm}_{({\rm II})}$. We show that among these five $\mathcal{R}$ matrices only the $\mathcal{R}^{\pm}_{({\rm I})}$ are consistent with the commutation relations and the real structure of the quantum Minkowski space–time and the quantum Lorentz group.

2. Bicovariant calculus on *SL*(2, *C*) and *SU*(2) quantum groups

Before we start to construct the quantum Lorentz group out of the quantum $SL(2, C)$ group, let us recall some results provided by the bicovariant calculus over the $SL(2, C)$ and $SU(2)$ quantum groups. Let a unital $\star$-algebra $\mathcal{A}$ be generated by $M_\alpha^\beta (\alpha, \beta = 1, 2)$ which preserves a nondegenerate bilinear from ε

$$\varepsilon_{\alpha\beta}M_\gamma{}^\alpha M_\delta{}^\beta=\varepsilon_{\gamma\delta}I_{\mathcal{A}},\quad \varepsilon^{\gamma\delta}M_\gamma{}^\alpha M_\delta{}^\beta=\varepsilon^{\alpha\beta}I_{\mathcal{A}},\quad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta}=\delta^\alpha_\beta=\varepsilon_{\beta\gamma}\varepsilon^{\gamma\alpha},$$

which are the unimodularity conditions. The nondegenerate bilinear form ε is considered as a quantum spinor metric, $I_{\mathcal{A}}$ being the unity of $\mathcal{A}$. To preserve these conditions under the antimultiplicative involution $\star:\mathcal{A}\to\mathcal{A}$, the spinor metric must satisfy the condition $(\varepsilon_{\alpha\beta})^\star=\lambda\varepsilon_{\dot\beta\dot\alpha}$ with $\lambda\lambda^\star=1$ leading to

$$\varepsilon_{\dot\alpha\dot\beta}M_{\dot\gamma}{}^{\dot\alpha}M_{\dot\delta}{}^{\dot\beta}=\varepsilon_{\dot\gamma\dot\delta}I_{\mathcal{A}},\quad \varepsilon^{\dot\gamma\dot\delta}M_{\dot\gamma}{}^{\dot\alpha}M_{\dot\delta}{}^{\dot\beta}=\varepsilon^{\dot\alpha\dot\beta}I_{\mathcal{A}},\quad \varepsilon^{\dot\alpha\dot\gamma}\varepsilon_{\dot\gamma\dot\beta}=\delta^{\dot\alpha}_{\dot\beta}=\varepsilon_{\dot\beta\dot\gamma}\varepsilon^{\dot\gamma\dot\alpha},$$

where $M_{\dot\alpha}{}^{\dot\beta}=(M_\alpha{}^\beta)^\star$. For convenience, we take $\lambda=1$. $\mathcal{A}$ carries a structure of a $\star$-Hopf algebra with a coaction $\Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$, a counit $\varepsilon:\mathcal{A}\to\mathcal{C}$ and an antipode $S:\mathcal{A}\to\mathcal{A}$ defined on the generators by $\Delta(M_\alpha{}^\beta)=M_\alpha{}^\gamma\otimes M_\gamma{}^\beta$, $\varepsilon(M_\alpha{}^\beta)=\delta^\beta_\alpha$ and $S(M_\alpha{}^\beta)=\varepsilon_{\alpha\gamma}M_\delta{}^\gamma\varepsilon^{\delta\beta}$. On the dotted copy, we have $(\Delta(M_\alpha{}^\beta))^\star=\Delta((M_\alpha{}^\beta)^\star)=\Delta(M_{\dot\alpha}{}^{\dot\beta})=M_{\dot\alpha}{}^{\dot\gamma}\otimes M_{\dot\gamma}{}^{\dot\beta}$, $\varepsilon(M_{\dot\alpha}{}^{\dot\beta})=\delta^{\dot\beta}_{\dot\alpha}$ and $S(M_{\dot\alpha}{}^{\dot\beta})=\varepsilon_{\dot\alpha\dot\gamma}M_{\dot\delta}{}^{\dot\gamma}\varepsilon^{\dot\delta\dot\beta}$. The involution $\star$ acts on the antipode as $(S(M_\alpha{}^\beta))^\star=\varepsilon^{\dot\beta\dot\delta}M_{\dot\delta}{}^{\dot\gamma}\varepsilon_{\dot\gamma\dot\alpha}=S^{-1}(M_{\dot\alpha}{}^{\dot\beta})$.

It is known [11] that the generators of such a system satisfy the noncommutativity relations $R^{\pm\alpha\beta}_{\sigma\rho}M_\gamma{}^\sigma M_\delta{}^\rho=M_\sigma{}^\alpha M_\rho{}^\beta R^{\pm\sigma\rho}_{\gamma\delta}$ where the forms of the R matrices are given by $R^{\pm\alpha\beta}_{\gamma\delta}=\delta^\alpha_\gamma\delta^\beta_\delta+a^{\pm1}\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}$ satisfying $R^{\pm\alpha\beta}_{\sigma\rho}R^{\mp\sigma\rho}_{\gamma\delta}=\delta^\alpha_\gamma\delta^\beta_\delta$ with $Q+\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta}=0$, $Q=a+a^{-1}$ and $a\neq0$. The R matrices satisfy the Yang–Baxter equations, the Hecke equations $(R^\pm+a^{\pm2})(R^\pm-1)$ and $\varepsilon_{\alpha\beta}R^{\pm\alpha\lambda}_{\sigma\gamma}R^{\pm\beta\rho}_{\lambda\delta}=a^{\mp1}\varepsilon_{\gamma\delta}\delta^\rho_\sigma$.

Now, we consider a right-invariant basis θ_α of the bicovariant bimodule Γ over $\mathcal{A}$ on which the right coaction acts as $\Delta_R(\theta_\alpha)=\theta_\alpha\otimes I$, $\Delta_R(\theta_{\dot\alpha})=\theta_{\dot\alpha}\otimes I$ and the left coaction acts as

$$\begin{aligned}&\Delta_L(\theta_\alpha)=M_\alpha{}^\beta\otimes\theta_\beta,\quad \Delta_L(\theta^\alpha)=S(M_\beta{}^\alpha)\otimes\theta^\beta,\qquad \Delta_L(\theta_{\dot\alpha})=M_{\dot\alpha}{}^{\dot\beta}\otimes\theta_{\dot\beta},\\ &\Delta_L(\theta^{\dot\alpha})=S^{-1}(M_{\dot\beta}{}^{\dot\alpha})\otimes\theta^{\dot\beta},\end{aligned}\tag{1}$$

where $(\theta_\alpha)^\star=\theta_{\dot\alpha}$ and the spinorial indices are lowered and raised as $\theta_\alpha=\theta^\beta\varepsilon_{\beta\alpha}$, $\theta^\alpha=\theta_\beta\varepsilon^{\beta\alpha}$, $\theta^{\dot\alpha}=\varepsilon^{\dot\alpha\dot\beta}\theta_{\dot\beta}$ and $\theta_{\dot\alpha}=\varepsilon_{\dot\alpha\dot\beta}\theta^{\dot\beta}$. From the bicovariance properties of the bimodule $\mathcal{A}$–Γ [4] we can show the existence of functionals $f:\mathcal{A}\to C$ satisfying the following properties:

$$\theta_\alpha a=(a\star f_\alpha{}^\beta)\theta_\beta,\quad \theta^\alpha a=(a\star\tilde f_\beta{}^\alpha)\theta^\beta,\tag{2}$$

$$\theta_{\dot\alpha}a=(a\star f_{\dot\alpha}{}^{\dot\beta})\theta_{\dot\beta},\quad \theta^{\dot\alpha}a=(a\star\tilde f_{\dot\beta}{}^{\dot\alpha})\theta^{\dot\beta},\tag{3}$$

$$a\theta_\alpha=\theta_\beta(a\star f_\alpha{}^\beta\circ S),\quad a\theta^\alpha=\theta^\beta(a\star\tilde f_\beta{}^\alpha\circ S),\tag{4}$$

$$a\theta_{\dot\alpha}=\theta_{\dot\beta}(a\star f_{\dot\alpha}{}^{\dot\beta}\circ S),\quad a\theta^{\dot\alpha}=\theta^{\dot\beta}(a\star\tilde f_{\dot\beta}{}^{\dot\alpha}\circ S),\tag{5}$$

$$f^\beta_\alpha(ab)=f^\gamma_\alpha(a)f^\beta_\gamma(b),\quad \tilde f_\alpha{}^\beta(ab)=\tilde f_\gamma{}^\beta(a)\tilde f_\alpha{}^\gamma(b),\tag{6}$$

$$f_{\dot\alpha}{}^{\dot\beta}(ab)=f_{\dot\alpha}{}^{\dot\gamma}(a)f_{\dot\gamma}{}^{\dot\beta}(b),\quad \tilde f_{\dot\alpha}{}^{\dot\beta}(ab)=\tilde f_{\dot\gamma}{}^{\dot\beta}(a)\tilde f_{\dot\alpha}{}^{\dot\gamma}(b),\tag{7}$$

$$f_\alpha{}^\beta(I)=\delta_\alpha^\beta=\tilde f_\alpha{}^\beta(I),\qquad f_{\dot\alpha}{}^{\dot\beta}(I)=\delta_{\dot\alpha}^{\dot\beta}=\tilde f_{\dot\alpha}{}^{\dot\beta}(I), \tag{8}$$

$$M_\alpha{}^\gamma(f_\gamma{}^\beta\star a)=(a\star f_\alpha{}^\gamma)M_\gamma{}^\beta,\quad S(M_\gamma{}^\alpha)(\tilde f_\beta{}^\gamma\star a)=(a\star\tilde f_\gamma{}^\alpha)S(M_\beta{}^\gamma), \tag{9}$$

$$M_{\dot\alpha}{}^{\dot\gamma}(f_{\dot\gamma}{}^{\dot\beta}\star a)=(a\star f_{\dot\alpha}{}^{\dot\gamma})M_{\dot\gamma}{}^{\dot\beta},\ \ S^{-1}(M_{\dot\gamma}{}^{\dot\alpha})(\tilde f_{\dot\beta}{}^{\dot\gamma}\star a)=(a\star\tilde f_{\dot\gamma}{}^{\dot\alpha})S^{-1}(M_{\dot\beta}{}^{\dot\gamma}), \tag{10}$$

where the convolution product is defined by $a\star f=(f\otimes I)\Delta(a)$ and $f\star a=(I\otimes f)\Delta(a)$ for any $a\in\mathcal{A}$. Setting $a=S(a)(S^{-1}(a))$ into the right undotted (dotted) relation of (9) and (10), then applying $S^{-1}(S)$ and comparing with the corresponding left undotted (dotted) relation of (9) and (10), we get

$$f_\alpha{}^\beta=\tilde f_\alpha{}^\beta\circ S,\qquad f_{\dot\alpha}{}^{\dot\beta}=\tilde f_{\dot\alpha}{}^{\dot\beta}\circ S^{-1}. \tag{11}$$

For the generators of $\mathcal{A}$, the left relation (9) gives $M_\alpha{}^\gamma M_\rho{}^\delta f_\gamma{}^\beta(M_\delta{}^\sigma)=f_\alpha{}^\gamma=(M_\rho{}^\delta)M_\delta{}^\sigma M_\gamma{}^\beta$ which shows that there exist two functionals $f_{\pm\gamma}{}^\alpha(M_\delta{}^\rho)$ proportional to the $R^{\pm\rho\alpha}_{\gamma\delta}$ matrices. Applying these functionals on both sides of the unimodularity condition, we obtain: $f_{\pm\gamma}{}^\alpha(M_\delta{}^\rho)=a^{\mp(1/2)}R^{\pm\rho\alpha}_{\gamma\delta}$ and $f_{\pm\gamma}{}^\alpha(S(M_\delta{}^\rho))=a^{\pm(1/2)}R^{\mp\alpha\gamma}_{\delta\gamma}$ [12]. The same procedure can be used for the dotted copy of $\mathcal{A}$ generators. Then there exists two bases $\theta_{\pm\alpha}$ corresponding to the functionals $f_{\pm\alpha}{}^\beta$.

Applying the $\star$ involution on both sides of (2) and (3), we get respectively

$$\begin{aligned}
(\theta_{\pm\alpha}a)^\star&=(\theta_{\pm\beta})^\star(f_{\pm\alpha}{}^\beta(a_{(1)}))^\star a^\star_{(2)}=a^\star(\theta_{\pm\alpha})^\star,\\
(\theta^\alpha_\pm a)^\star&=(\theta^\beta_\pm)^\star(\tilde f_{\pm\beta}{}^\alpha(a_{(1)}))^\star a^\star_{(2)}=a^\star(\theta^\alpha_\pm)^\star,\\
(\theta_{\pm\dot\alpha}a)^\star&=(\theta_{\pm\dot\beta})^\star(f_{\pm\dot\alpha}{}^{\dot\beta}(a_{(1)}))^\star a^\star_{(2)}=a^\star(\theta_{\pm\dot\alpha})^\star,\\
(\theta^{\dot\alpha}_\pm a)^\star&=(\theta^{\dot\beta}_\pm)^\star(\tilde f_{\pm\dot\beta}{}^{\dot\alpha}(a_{(1)}))^\star a^\star_{(2)}=a^\star(\theta^{\dot\alpha}_\pm)^\star,
\end{aligned}\tag{12}$$

where $a_{(1)}$ and $a_{(2)}$ denote elements of $\Delta(a)=a_{(1)}\otimes a_{(2)}$. On the other hand, for $a=a^\star$, (4) and (5) give respectively

$$\begin{aligned}
a^\star\theta_{\pm\alpha}&=\theta_{\pm\beta}f_{\pm\alpha}{}^\beta(S(a^\star_{(1)}))a^\star_{(2)},\qquad a^\star\theta^\alpha_\pm=\theta^\beta_\pm\tilde f_{\pm\beta}{}^\alpha(S(a^\star_{(1)}))a^\star_{(2)},\\
a^\star\theta_{\pm\dot\alpha}&=\theta_{\pm\dot\beta}f_{\pm\dot\alpha}{}^{\dot\beta}(S(a^\star_{(1)}))a^\star_{(2)},\qquad a^\star\theta^{\dot\alpha}_\pm=\theta^{\dot\beta}_\pm\tilde f_{\pm\dot\beta}{}^{\dot\alpha}(S(a^\star_{(1)}))a^\star_{(2)}.
\end{aligned}\tag{13}$$

But for $a=M_\sigma{}^\rho$, we have $(f_{\pm\alpha}{}^\beta(M_\sigma{}^\delta))^\star=(a^{\mp(1/2)}R^{\pm\delta\beta}_{\alpha\sigma})^\star=a^{\mp(1/2)}R^{\pm\dot\beta\dot\delta}_{\dot\sigma\dot\alpha}=f_{\mp\dot\alpha}{}^{\dot\beta}(S(M_{\dot\sigma}{}^{\dot\delta}))$ for a real. Therefore (12) and (13) are consistent if $(\theta_{\pm\alpha})^\star=\theta_{\mp\dot\alpha}$ and $(\theta^\alpha_\pm)^\star=\theta^{\dot\alpha}_\mp$ yielding

$$\begin{aligned}
(f_{\pm\alpha}{}^\beta(a))^\star&=f_{\mp\dot\alpha}{}^{\dot\beta}(S(a^\star)),\qquad (\tilde f_{\pm\alpha}{}^\beta(a))^\star=\tilde f_{\mp\dot\alpha}{}^{\dot\beta}(S(a^\star)),\\
(f_{\pm\dot\alpha}{}^{\dot\beta}(a))^\star&=f_{\mp\alpha}{}^\beta(S(a^\star)),\qquad (\tilde f_{\pm\dot\alpha}{}^{\dot\beta}(a))^\star=\tilde f_{\mp\alpha}{}^\beta(S(a^\star)).
\end{aligned}\tag{14}$$

Finally, by using the spinor metric to raise the indices of the right invariant bases in (2) and (3), we may also show that

$$f_{\pm\alpha}{}^\beta=\varepsilon^{\beta\delta}\tilde f_{\pm\delta}{}^\gamma\varepsilon_{\gamma\alpha}\quad\text{and}\quad\varepsilon^{\dot\beta\dot\delta}f_{\pm\dot\delta}{}^{\dot\gamma}\varepsilon_{\dot\gamma\dot\alpha}=\tilde f_{\pm\dot\alpha}{}^{\dot\beta}. \tag{15}$$

In this stage, we have no indication on the explicit forms of $f_{\pm\gamma}{}^\alpha(M_{\dot\delta}{}^{\dot\rho})$ or $f_{\pm\dot\gamma}{}^{\dot\alpha}(M_\delta{}^\rho)$ to control the noncommutativity between undotted and dotted generators of the quantum

$SL(2,C)$ group. To carry this point we assume either the generators $M_\alpha{}^\beta$ commute with $M_{\dot\alpha}{}^{\dot\beta}$ or are controlled by the R matrices satisfying the properties of the quantum $SU(2)$ group. In the following we assume the latter possibility.

To reflect the specific properties of the quantum $SU(2)$ group, we have to add the unitarity condition on the generators as $M_{\dot\alpha}{}^{\dot\beta} = S(M_\beta{}^\alpha)$ and $M_\alpha{}^\beta = S^{-1}(M_{\dot\beta}{}^{\dot\alpha})$. Applying S on both sides of the unimodularity condition, we may show that the unitarity condition yields $\varepsilon_{\alpha\beta} = \lambda\varepsilon^{\dot\beta\dot\alpha}$ with $\lambda\lambda^\star = 1$. In the following we take $\lambda = -1$. Then to be consistent with the quantum $SU(2)$ group, the spinor metric must satisfy

$$(\varepsilon_{\alpha\beta})^\star = \varepsilon_{\dot\beta\dot\alpha} = -\varepsilon^{\alpha\beta} \quad \text{and} \qquad (\varepsilon^{\alpha\beta})^\star = \varepsilon^{\dot\beta\dot\alpha} = -\varepsilon_{\alpha\beta}. \tag{16}$$

It is easy to see, by using the unitarity condition of the generators into (1), that $\theta_{\dot\alpha} = \theta^\alpha$ and $\theta_\alpha = \theta^{\dot\alpha}$ implying conditions on the functionals

$$f_{\pm\dot\alpha}{}^{\dot\beta} = \tilde f_{\pm\beta}{}^\alpha, \quad \tilde f_{\pm\dot\alpha}{}^{\dot\beta} = f_{\pm\beta}{}^\alpha. \tag{17}$$

As stated above, we assume that the functionals of the quantum $SL(2,C)$ group satisfy the same properties as the $SU(2)$ ones. Therefore, if for example we set $a = M_{\dot\sigma}{}^{\dot\rho}$ into the left relation of (2), we get

$$M_\alpha{}^\gamma M_{\dot\sigma}{}^{\dot\delta} f_{\pm\gamma}{}^\beta (M_{\dot\delta}{}^{\dot\rho}) = f_{\pm\alpha}{}^\gamma (M_{\dot\sigma}{}^{\dot\delta}) M_{\dot\delta}{}^{\dot\rho} M_\gamma{}^\beta \quad \text{or}$$
$$M_\alpha{}^\gamma M_{\dot\sigma}{}^{\dot\delta} R^{\pm\dot\rho\beta}_{\gamma\dot\delta} = R^{\pm\dot\delta\gamma}_{\alpha\dot\sigma} M_{\dot\delta}{}^{\dot\rho} M_\gamma{}^\beta, \tag{18}$$

where $f_{\pm\alpha}{}^\gamma (M_{\dot\sigma}{}^{\dot\delta}) = R^{\pm\dot\delta\gamma}_{\alpha\dot\sigma} = f_{\pm\alpha}{}^\gamma (S(M_\delta{}^\sigma)) = a^{\pm(1/2)} R^{\mp\gamma\sigma}_{\delta\alpha}$. We have also objects of the form

$$f_{\pm\alpha}{}^\gamma (S(M_{\dot\sigma}{}^{\dot\delta})) = R^{\mp\gamma\dot\delta}_{\dot\sigma\alpha} = f_{\pm\alpha}{}^\gamma (S(S(M_\delta{}^\sigma))) = \varepsilon_{\delta\lambda} f_{\pm\alpha}{}^\gamma (S(M_\nu{}^\lambda))\varepsilon^{\nu\sigma}$$
$$= a^{\pm(1/2)}\varepsilon_{\delta\lambda} R^{\mp\gamma\lambda}_{\nu\alpha}\varepsilon^{\nu\sigma} = \varepsilon_{\dot\sigma\dot\rho} f_{\pm\alpha}{}^\gamma (M_{\dot\gamma}{}^{\dot\rho})\varepsilon^{\dot\gamma\dot\delta} = \varepsilon_{\dot\sigma\dot\rho} R^{\pm\dot\rho\gamma}_{\alpha\dot\lambda}\varepsilon^{\dot\lambda\dot\delta}.$$

From (6) and (7), we get respectively

$$f_{\pm\alpha}{}^\gamma (M_{\dot\sigma}{}^{\dot\delta}) f_{\pm\gamma}{}^\beta (S(M^{\dot\rho}_{\dot\delta})) = \delta^\beta_\alpha \delta^{\dot\rho}_{\dot\sigma} = R^{\pm\dot\delta\gamma}_{\alpha\dot\sigma} R^{\mp\beta\dot\rho}_{\dot\delta\gamma}, \tag{19}$$

$$f_{\pm\alpha}{}^\gamma (S(M^{\dot\delta}_{\dot\sigma})) f_{\pm\gamma}{}^\beta (M_{\dot\delta}{}^{\dot\rho}) = \delta^\beta_\alpha \delta^{\dot\rho}_{\dot\sigma} = R^{\mp\gamma\dot\delta}_{\dot\sigma\alpha} R^{\pm\dot\rho\beta}_{\gamma\dot\delta}. \tag{20}$$

3. The quantum Lorentz group

To have a correspondence between $SL(2,C)$ and Lorentz quantum groups, we must construct the quantum analog of the Pauli matrices. Let us consider an element $X_{\alpha\dot\beta}$ as a tensor product of an undotted and dotted elements of right invariant basis of the bimodule $\mathcal{A}$–Γ (bispinor). $X_{\alpha\dot\beta}$ can always be expanded on a system of four independent 2×2 matrices $\sigma^I_{\alpha\dot\beta} (I = 0, \ldots, 3)$ as $X_I\sigma^I_{\alpha\dot\beta}$, $X_{\alpha\dot\beta}$ tranforms as

$$\Delta_L(X_{\alpha\dot\beta}) = M_\alpha{}^\sigma M_{\dot\beta}{}^{\dot\rho} \otimes X_{\sigma\dot\rho}, \quad \Delta_R(X_{\alpha\dot\beta}) = X_{\alpha\dot\beta} \otimes I. \tag{21}$$

Then, we have

Proposition 3.1.

1. *There exist four* 2×2 *matrices* $\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta}$ *given by*

$$\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\lambda}} R^{\mp\sigma\dot{\rho}}_{\dot{\lambda}\nu} \varepsilon^{\nu\beta} \sigma^{I}_{\sigma\dot{\rho}} \tag{22}$$

such that $X_I\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta} = X_{\pm}{}^{\dot{\alpha}\beta}$ *trans forms under the left coaction as*

$$\Delta_L(X_{\pm}{}^{\dot{\alpha}\beta}) = S^{-1}(M_{\dot{\sigma}}{}^{\dot{\alpha}})S(M_{\rho}{}^{\beta}) \otimes X_{\pm}{}^{\dot{\sigma}\rho}.$$

2. $\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta}$ *are hermitian iff* $\sigma^{I}_{\alpha\dot{\beta}}$ *are.*

Proof.

1. Setting into (21) $M_{\alpha}{}^{\sigma} M_{\dot{\beta}}{}^{\dot{\rho}} = R^{\pm\dot{\gamma}\delta}_{\alpha\dot{\beta}} M_{\dot{\gamma}}{}^{\dot{\mu}} M_{\delta}{}^{\nu} R^{\mp\sigma\dot{\rho}}_{\dot{\mu}\nu}$, obtained by multiplying from the right both sides of (18) by $R^{\mp\xi\dot{\tau}}_{\dot{\rho}\beta}$ and by using (19), we obtain

$$\Delta_L(X_{\alpha\dot{\beta}}) = R^{\pm\dot{\gamma}\delta}_{\alpha\dot{\beta}} M_{\dot{\gamma}}{}^{\dot{\mu}} M_{\delta}{}^{\nu} R^{\mp\sigma\dot{\rho}}_{\dot{\mu}\nu} \otimes X_{\sigma\dot{\rho}}.$$

Multiplying from the left both sides by $R^{\mp\alpha\dot{\beta}}_{\dot{\lambda}\nu}$ and using (20), we deduce

$$\begin{aligned}\Delta_L(X_I R^{\pm\alpha\dot{\beta}}_{\dot{\lambda}\nu}\sigma^{I}_{\alpha\dot{\beta}}) &= M_{\dot{\lambda}}{}^{\dot{\mu}} M_{\nu}{}^{\tau} \otimes X_I R^{\mp\sigma\dot{\rho}}_{\dot{\mu}\tau}\sigma^{I}_{\sigma\dot{\rho}}\\ &= \varepsilon_{\dot{\lambda}\dot{\gamma}} S^{-1}(M_{\dot{\delta}}{}^{\dot{\gamma}})\varepsilon^{\dot{\delta}\dot{\mu}}\varepsilon^{\tau\xi} S(M_{\xi}{}^{\kappa})\varepsilon_{\kappa\nu} \otimes X_I R^{\mp\sigma\dot{\rho}}_{\dot{\mu}\tau}\sigma^{I}_{\sigma\dot{\rho}}\end{aligned}$$

yielding

$$\Delta_L(X_I\varepsilon^{\dot{\alpha}\dot{\lambda}} R^{\pm\sigma\dot{\rho}}_{\dot{\lambda}\nu}\varepsilon^{\nu\beta}\sigma^{I}_{\sigma\dot{\rho}}) = S^{-1}(M_{\dot{\delta}}{}^{\dot{\alpha}})S(M_{\gamma}{}^{\beta}) \otimes X_I\varepsilon^{\dot{\delta}\dot{\lambda}} R^{\mp\sigma\dot{\rho}}_{\dot{\lambda}\nu}\varepsilon^{\nu\gamma}\sigma^{I}_{\sigma\dot{\rho}},$$

which can be written in the form $\Delta_L(X_{\pm}^{\dot{\alpha}\beta}) = \Delta_L(X_I\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta}) = S^{-1}(M_{\dot{\delta}}{}^{\dot{\alpha}})S(M_{\gamma}{}^{\beta}) \otimes X_{\pm}{}^{\dot{\delta}\gamma}$ with

$$\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\lambda}} R^{\mp\sigma\dot{\rho}}_{\dot{\lambda}\nu}\varepsilon^{\nu\beta}\sigma^{I}_{\sigma\dot{\rho}}$$

from which we obtain $\varepsilon_{\dot{\lambda}\dot{\alpha}}\bar{\sigma}_{\pm}^{I\dot{\alpha}\beta}\varepsilon_{\beta\nu} = R^{\mp\sigma\dot{\rho}}_{\dot{\lambda}\nu}\sigma^{I}_{\sigma\dot{\rho}}$. Multiplying from the right both sides by $R^{\pm\dot{\lambda}\nu}_{\gamma\dot{\tau}}$ and using (19), we get

$$\sigma^{I}_{\alpha\dot{\beta}} = \varepsilon_{\dot{\lambda}\dot{\gamma}} R^{\pm\dot{\lambda}\nu}_{\alpha\dot{\beta}}\varepsilon_{\mu\nu}\bar{\sigma}_{\pm}^{I\dot{\gamma}\mu}. \tag{23}$$

2. Under the conditions (16) on the spinor metric we have $(R^{\pm\alpha\rho}_{\sigma\nu})^{\star} = R^{\pm\sigma\nu}_{\alpha\rho}$, for a real, implying

$$(R^{\pm\alpha\dot{\beta}}_{\dot{\lambda}\dot{\nu}})^{\star} = (a^{\pm(1/2)}\varepsilon_{\beta\rho} R^{\pm\alpha\rho}_{\sigma\nu}\varepsilon^{\sigma\lambda})^{\star} = a^{\pm(1/2)}\varepsilon^{\beta\rho} R^{\pm\sigma\nu}_{\alpha\rho}\varepsilon_{\sigma\lambda}.$$

On the other hand, by using (11) and (15), we obtain

$$a^{\pm(1/2)}\varepsilon^{\beta\rho}R^{\mp\sigma\nu}_{\alpha\rho}\varepsilon_{\sigma\lambda}=f_{\mp\alpha}{}^{\nu}(\varepsilon^{\beta\rho}M_{\rho}{}^{\sigma}\varepsilon_{\sigma\lambda})=f_{\pm\alpha}{}^{\nu}(S^{-1}(M_{\lambda}{}^{\beta}))=\tilde{f}_{\mp\alpha}{}^{\nu}(M_{\lambda}{}^{\beta})$$
$$=\varepsilon_{\alpha\sigma}f_{\mp\delta}{}^{\sigma}(M_{\lambda}{}^{\beta})\varepsilon^{\delta\nu}=a^{\pm(1/2)}\varepsilon_{\alpha\sigma}R^{\mp\beta\sigma}_{\delta\lambda}\varepsilon^{\delta\nu}=R^{\mp\beta\dot{\alpha}}_{\dot{\nu}\lambda}$$

yielding $(\bar{\sigma}^{I\dot{\alpha}\beta}_{\pm})^{\star}=\varepsilon^{\dot{\beta}\dot{\nu}}R^{\mp\rho\dot{\sigma}}_{\dot{\nu}\lambda}\varepsilon^{\lambda\alpha}(\sigma^{I}_{\sigma\dot{\rho}})^{\star}$. Therefore, if $(\sigma^{I}_{\sigma\dot{\rho}})^{\star}=\sigma^{I}_{\rho\dot{\sigma}}$ then

$$(\bar{\sigma}^{I\dot{\alpha}\beta}_{\pm})^{\star}=\varepsilon^{\dot{\beta}\dot{\nu}}R^{\mp\rho\dot{\sigma}}_{\dot{\nu}\lambda}\varepsilon^{\lambda\alpha}\sigma^{I}_{\rho\dot{\sigma}}=\bar{\sigma}^{I\dot{\beta}\alpha}_{\pm}.$$

The same procedure can be applied to (23) to show the converse. □

To define a quantum metric of the space $\mathcal{M}$ spanned by X_I, we have to define an adequate trace [12,13,19] over the spinorial indices which makes invariant this metric under quantum $SL(2,C)$ group.

Proposition 3.2. *$\mathcal{M}$ is endowed with a metric G^{IJ} given by*

$$G_{\pm}{}^{IJ}=\frac{1}{Q}\mathrm{Tr}(\sigma^{I}\bar{\sigma}^{J}_{\pm})=\frac{1}{Q}\varepsilon^{\alpha\nu}\sigma^{I}_{\alpha\dot{\beta}}\bar{\sigma}^{J\dot{\beta}\gamma}_{\pm}\varepsilon_{\gamma\nu}=\frac{1}{Q}\mathrm{Tr}(\bar{\sigma}^{I}_{\pm}\sigma^{J})=\frac{1}{Q}\varepsilon_{\dot{\nu}\dot{\gamma}}\bar{\sigma}^{I\dot{\gamma}\alpha}_{\pm}\sigma^{J}_{\alpha\dot{\beta}}\varepsilon^{\dot{\nu}\dot{\beta}} \tag{24}$$

such that

1. *$G_{\pm}{}^{IJ}X_IX_J$ are invariant under quantum SL(2, C) group.*
2. *$G_{\pm}{}^{IJ}$ are hermitian if the matrices $\sigma^{I}_{\alpha\dot{\beta}}$ are.*

Proof.

1. To show the invariance of G under the quantum $SL(2,C)$ group, we consider the length of four-vector X_I, $G_{\pm}{}^{IJ}X_IX_J=(\varepsilon^{\alpha\nu}X_{\alpha\dot{\beta}}X_{\pm}{}^{\dot{\beta}\gamma}\varepsilon_{\gamma\nu})/Q$, which transforms under $SL(2,C)$ as

$$\begin{aligned}\Delta_L(G_{\pm}{}^{IJ}X_IX_J)&=\frac{1}{Q}\varepsilon^{\alpha\nu}M_{\alpha}{}^{\sigma}M_{\dot{\beta}}{}^{\dot{\rho}}S^{-1}(M_{\dot{\delta}}{}^{\dot{\beta}})S(M_{\lambda}{}^{\gamma})\varepsilon_{\gamma\nu}\otimes X_{\sigma\dot{\rho}}X_{\pm}{}^{\dot{\delta}\lambda}\\&=\frac{1}{Q}\varepsilon^{\alpha\nu}M_{\alpha}{}^{\sigma}S(M_{\lambda}{}^{\gamma})\varepsilon_{\gamma\nu}\otimes X_{\sigma\dot{\delta}}X_{\pm}{}^{\dot{\delta}\lambda}\\&=\frac{1}{Q}\varepsilon^{\alpha\nu}M_{\alpha}{}^{\sigma}\varepsilon_{\lambda\rho}M_{\mu}{}^{\rho}\varepsilon^{\mu\gamma}\varepsilon_{\gamma\nu}\otimes X_{\sigma\dot{\delta}}X_{\pm}{}^{\dot{\delta}\lambda}\\&=I\otimes\frac{1}{Q}\varepsilon^{\sigma\rho}X_{\sigma\dot{\delta}}X_{\pm}{}^{\dot{\delta}\lambda}\varepsilon_{\lambda\rho}=I\otimes\frac{1}{Q}\varepsilon^{\sigma\rho}\sigma^{I}_{\sigma\dot{\delta}}\bar{\sigma}^{J\dot{\delta}\lambda}_{\pm}\varepsilon_{\lambda\rho}X_IX_J\\&=I\otimes G_{\pm}{}^{IJ}X_IX_J.\end{aligned}$$

The same computation may be applied to show that $\varepsilon_{\dot{\nu}\dot{\gamma}}X_{\pm}{}^{\dot{\gamma}\alpha}X_{\alpha\dot{\beta}}\varepsilon^{\dot{\nu}\dot{\beta}}$ is invariant under quantum $SL(2,C)$ group. Now, using (22) and the form of the R matrices, we obtain

$$
\begin{aligned}
G_{\pm}{}^{IJ} &= \frac{1}{Q}\varepsilon^{\alpha\xi}\sigma^{I}_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\lambda}}R^{\mp\sigma\dot{\rho}}_{\dot{\lambda}\nu}\varepsilon^{\nu\gamma}\sigma^{J}_{\sigma\dot{\rho}}\varepsilon_{\gamma\xi} \\
&= \frac{1}{Q}a^{\pm(1/2)}\varepsilon^{\alpha\xi}\sigma^{I}_{\alpha\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\lambda}}\varepsilon_{\rho\delta}R^{\mp\sigma\delta}_{\mu\nu}\varepsilon^{\mu\lambda}\varepsilon^{\nu\gamma}\sigma^{J}_{\sigma\dot{\rho}}\varepsilon_{\gamma\xi} \\
&= \frac{1}{Q}a^{\pm(1/2)}\varepsilon^{\alpha\xi}\varepsilon^{\dot{\beta}\dot{\lambda}}\varepsilon_{\rho\delta}(\delta^{\sigma}_{\mu}\delta^{\delta}_{\xi}+a^{\mp1}\varepsilon^{\sigma\delta}\varepsilon_{\mu\xi})\varepsilon^{\mu\lambda}\sigma^{I}_{\alpha\dot{\beta}}\sigma^{J}_{\sigma\dot{\rho}} \\
&= -\frac{1}{Q}a^{\pm(1/2)}\varepsilon^{\alpha\xi}\varepsilon^{\dot{\beta}\dot{\lambda}}\varepsilon_{\rho\delta}(\delta^{\sigma}_{\mu}\delta^{\delta}_{\xi}+a^{\mp1}\varepsilon^{\sigma\delta}\varepsilon_{\mu\xi})\varepsilon_{\dot{\lambda}\dot{\mu}}\sigma^{I}_{\alpha\dot{\beta}}\sigma^{J}_{\sigma\dot{\rho}} \\
&= -\frac{1}{Q}a^{\pm(1/2)}\varepsilon^{\alpha\xi}\varepsilon_{\rho\delta}(\delta^{\sigma}_{\mu}\delta^{\delta}_{\xi}+a^{\mp1}\varepsilon^{\sigma\delta}\varepsilon_{\mu\xi})\sigma^{I}_{\alpha\dot{\mu}}\sigma^{J}_{\sigma\dot{\rho}} \\
&= \frac{1}{Q}(a^{\pm(1/2)}\sigma^{I\xi}_{\dot{\mu}}\sigma^{J\dot{\xi}}_{\mu}-a^{\mp12}\sigma^{I\xi\dot{\xi}}\sigma^{J\delta\dot{\delta}}),
\end{aligned}
\tag{25}
$$

where we have used (16) in the third and fifth line and the σ^I indices are raised and lowered as for the basis of the bicovariant $\mathcal{A}$–Γ bimodule ($\sigma^{I\alpha}_{\dot{\beta}} = \sigma^{I}_{\rho\dot{\beta}}\varepsilon^{\rho\alpha}$, $\sigma^{I\dot{\beta}}_{\alpha} = \varepsilon^{\dot{\beta}\dot{\rho}}\sigma^{I}_{\alpha\dot{\rho}}$ etc, ..). From a similar computation we can show that $\mathrm{Tr}(\bar{\sigma}^{I}_{\pm}\sigma^{J})$ gives the same form (25) for $G^{IJ}_{\pm}$.

2. if $\sigma^{I}_{\alpha\dot{\beta}}$ are hermitian, we have from (16) and the Proposition 3.1

$$
(G_{\pm}{}^{IJ})^{\star} = \frac{1}{Q}(\varepsilon^{\alpha\nu}\sigma^{I}_{\alpha\dot{\beta}}\bar{\sigma}^{J\dot{\beta}\gamma}_{\pm}\varepsilon_{\gamma\nu})^{\star} = \frac{1}{Q}\varepsilon_{\dot{\nu}\dot{\gamma}}\bar{\sigma}^{J\dot{\gamma}\beta}_{\pm}\sigma^{I}_{\beta\dot{\alpha}}\varepsilon^{\dot{\nu}\dot{\alpha}} = G_{\pm}{}^{JI}.
$$

which shows that the metric $G_{\pm}{}^{IJ}$ is hermitian.

□

Now, we can give an explicit example where $\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} = \varepsilon_{\dot{\beta}\dot{\alpha}} = -\varepsilon^{\dot{\beta}\dot{\alpha}} = \begin{pmatrix} 0 & -q^{-(1/2)} \\ q^{(1/2)} & 0 \end{pmatrix}$ and σ^I the usual four matrices which are the 2×2 identity matrix $\sigma^0_{\alpha\dot{\beta}}$ and the three Pauli matrices $\sigma^i_{\alpha\dot{\beta}}(i=1,2,3)$ as

$$
\sigma^0_{\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1_{\alpha\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2_{\alpha\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
$$
$$
\sigma^3_{\alpha\dot{\beta}} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1}, \end{pmatrix},
$$

where $q \neq 0$ is a real number, and $Q = a + a^{-1} = q + q^{-1} = -\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}$. With this choice, the metrics $G_{\pm}{}^{IJ}$ are of the form

$$
\begin{pmatrix}
-q^{\mp(3/2)} & 0 & 0 & 0 \\
0 & q^{\pm(1/2)} & -\mathrm{i}q^{\pm(1/2)}\dfrac{(q-q^{-1})}{Q} & 0 \\
0 & \mathrm{i}q^{\pm(1/2)}\dfrac{(q-q^{-1})}{Q} & q^{\pm(1/2)} & 0 \\
0 & 0 & 0 & q^{\pm(1/2)}
\end{pmatrix}.
$$

The inverse is given by

$$G_{\pm IJ}=\begin{pmatrix} -q^{\pm(3/2)} & 0 & 0 & 0\\ 0 & q^{\mp(1/2)}\dfrac{Q^2}{4} & -\mathrm{i}q^{\mp(1/2)}\dfrac{(q-q^{-1})Q}{4} & 0\\ 0 & \mathrm{i}q^{\mp(1/2)}\dfrac{(q-q^{-1})Q}{4} & q^{\mp(1/2)}\dfrac{Q^2}{4} & 0\\ 0 & 0 & 0 & q^{\mp(1/2)} \end{pmatrix}.$$

In the classical limit $q=1$, these metrics reduce to the classical Minkowski metric with signature $(-,+,+,+)$.

From a straightforward computation, we obtain the completeness relations as

$$\sigma^I_{\alpha\dot\beta}\bar\sigma_I{}^{\dot\rho\sigma}=Q\delta^\sigma_\alpha\varepsilon_{\dot\beta\dot\delta}\varepsilon^{\dot\rho\dot\delta},\quad \sigma_{\pm I\alpha\dot\beta}\bar\sigma_\pm^{I\dot\rho\sigma}=Q\delta^{\dot\rho}_{\dot\beta}\varepsilon_{\delta\alpha}\varepsilon^{\delta\sigma} \tag{26}$$

or

$$\sigma^{I\ \dot\beta}_\alpha\bar\sigma_{I\dot\rho}{}^\sigma=Q\delta^\sigma_\alpha\delta^{\dot\beta}_{\dot\rho},\quad \sigma_{\pm I\ \dot\beta}{}^\alpha\bar\sigma^{I\ \dot\rho}_{\pm\sigma}=Q\delta^\alpha_\sigma\delta^{\dot\rho}_{\dot\beta}, \tag{27}$$

where $\sigma_{\pm I\alpha\dot\beta}=G_{\pm IJ}\sigma^J_{\alpha\dot\beta}$. Note that a straightforward check shows that $\bar\sigma_J^{\dot\alpha\beta}=G_{+IJ}\bar\sigma_+^{J\dot\alpha\beta}=G_{-IJ}\bar\sigma_-^{J\dot\alpha\beta}=G_{\pm IJ}\bar\sigma_\pm^{J\dot\alpha\beta}$ and $\bar\sigma_+^{J\dot\alpha\beta}G_{+JI}=\bar\sigma_-^{J\dot\alpha\beta}G_{-JI}$.

Remark 3.1.

- *We may take as spinorial metrics satisfying (16), the most general form as*

$$\varepsilon_{\alpha\beta}=(\varepsilon_{\dot\beta\dot\alpha})^\star=d^{-(1/2)}\begin{pmatrix}\mathrm{i}r & -q^{-(1/2)}\\ q^{(1/2)} & \mathrm{i}r\end{pmatrix},$$
$$\varepsilon^{\alpha\beta}=(\varepsilon^{\dot\beta\dot\alpha})^\star=d^{-(1/2)}\begin{pmatrix}\mathrm{i}r & q^{-(1/2)}\\ -q^{(1/2)} & ir\end{pmatrix},$$

  where $q\neq 0$ and $r\neq\pm1$ are two real deformation parameters, $d=-r^2+1$ and $Q=a+a^{-1}=d^{-1}(2r+q+q^{-1})=-\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}$. From the computer MAPLE program we can show that the metrics $G^{IJ}_\pm$ are invertible and the completeness relations (26) and (27) remain valid.
- *The metric $G_{\pm IJ}$ can be written as $G_{\pm IJ}=G_{\pm IL}G_{\pm JK}G^{KL}_\pm=G_{\pm IL}G_{\pm JK}\frac{1}{Q}$ $\mathrm{Tr}(\sigma^K\bar\sigma^L_\pm)=\frac{1}{Q}\mathrm{Tr}(\sigma_{\pm J}\bar\sigma_I)=G_{\pm IL}G_{\pm JK}\frac{1}{Q}\mathrm{Tr}(\bar\sigma^K_\pm\sigma^L)=\frac{1}{Q}\mathrm{Tr}(\bar\sigma_J\sigma_{\pm I}$. With a way quite analogous that used to derive (25), we may show that*

$$G_{\pm IJ}=\frac{1}{Q}(a^{\mp(1/2)}\bar\sigma_{I\ \delta}{}^{\dot\beta}\bar\sigma_{J\dot\delta}{}^\beta-a^{\pm(1/2)}\bar\sigma_{I\dot\delta\delta}\bar\sigma_{J\dot\beta\beta}). \tag{28}$$

- *The completeness relations (27) may be used to convert a vector to a bispinor and vice versa*

$$X_{\alpha\dot\beta}=X_I\sigma^I_{\alpha\dot\beta}\Leftrightarrow X_I=\frac{1}{Q}\varepsilon^{\alpha\nu}X_{\alpha\dot\beta}\bar\sigma_\pm^{J\dot\beta\delta}\varepsilon_{\delta\nu}G_{\pm JI}\quad \textit{or}\quad X_I=\frac{1}{Q}\varepsilon_{\dot\nu\dot\beta}\bar\sigma_I{}^{\dot\beta\alpha}X_{\alpha\dot\delta}\varepsilon^{\dot\nu\dot\delta}.$$

- (21) *shows that* $X_{\alpha\dot{\beta}}$ *can be written in terms of spinor either as* $X_{\pm\alpha\dot{\beta}} = \theta_{\pm\alpha}\theta_{\mp\dot{\beta}}$ *or* ${X'}_{\pm\alpha\dot{\beta}} = \theta_{\pm\alpha}\theta_{\pm\dot{\beta}}$. *In the case where the* $\sigma^{I}_{\alpha\dot{\beta}}$ *matrices are hermitian, the former choice yields* $(X_{\pm\alpha\dot{\beta}})^{\star} = X_{\pm\beta\dot{\alpha}}$ *implying* $X_{\pm I}\sigma^{I}_{\beta\dot{\alpha}} = (X_{\pm I}\sigma^{I}_{\alpha\dot{\beta}})^{\star} = (X_{\pm I})^{\star}\sigma^{I}_{\beta\dot{\alpha}}$ *which shows that* $X_{\pm I}$ *are real. In this case the lengths* $G^{\pm IJ}X_{\pm I}X_{\pm J}$ *are also real.*

We are now ready to show how two copies of undotted and dotted generators of $\mathcal{A}$ may be combined to form generators ${\Lambda_L}^K$ of a unital algebra $\mathcal{L}$ corresponding to the quantum Lorentz group.

Theorem 3.1. *The generators* ${\Lambda_L}^K (L, K = 0, 1, 2, 3)$ *of the Quantum Lorentz group are given by*

$$ {\Lambda_L}^K = \frac{1}{Q}\varepsilon_{\dot{\gamma}\dot{\delta}}{\bar{\sigma}_L}^{\dot{\delta}\alpha}{M_\alpha}^{\sigma}\sigma^{K}_{\sigma\dot{\rho}}{M_{\dot{\beta}}}^{\dot{\rho}}\varepsilon^{\dot{\gamma}\dot{\beta}} = \frac{1}{Q}{\bar{\sigma}_{L\dot{\gamma}}}^{\alpha}{M_\alpha}^{\sigma}\sigma^{K\ \ \dot{\rho}}_{\sigma}S^{-1}({M_{\dot{\rho}}}^{\dot{\gamma}}) \tag{29}$$

or

$$ {\Lambda_L}^K = \frac{1}{Q}\varepsilon^{\alpha\delta}{M_\alpha}^{\sigma}\sigma^{K}_{\sigma\dot{\rho}}{M_{\dot{\beta}}}^{\dot{\rho}}\bar{\sigma}_{\pm}^{N\dot{\beta}\gamma}\varepsilon_{\gamma\delta}G_{\pm NL}. \tag{30}$$

They are real and satisfy the axiomatic structure of Hopf algebras with the relations

$$ G_{\pm NM}{\Lambda_L}^N{\Lambda_K}^M = G_{\pm LK}I_{\mathcal{L}} \quad \textit{and} \quad {G_{\pm}}^{LK}{\Lambda_L}^N{\Lambda_K}^M = {G_{\pm}}^{NM}I_{\mathcal{L}}, \tag{31}$$

where $I_{\mathcal{L}} = I_{\mathcal{A}}$ *is the unity of* $\mathcal{L} \subset \mathcal{A}$.

Proof. Multiplying from the left both sides of (21) by $\bar{\sigma}^{N\dot{\delta}\alpha}$ and making the trace over dotted indices, we get

$$ \Delta_L X_L \mathrm{Tr}(\bar{\sigma}_{\pm}^{N\dot{\delta}\alpha}\sigma^{L}_{\alpha\dot{\beta}}) = \Delta_L X_L Q G_{\pm}^{NL} = \varepsilon_{\dot{\gamma}\dot{\delta}}\bar{\sigma}_{\pm}^{N\dot{\delta}\alpha}{M_\alpha}^{\sigma}\sigma^{K}_{\sigma\dot{\rho}}{M_{\dot{\beta}}}^{\dot{\rho}}\varepsilon^{\dot{\gamma}\dot{\beta}} \otimes X_K $$

yielding

$$ \Delta_L X_L = \frac{1}{Q}\varepsilon_{\dot{\gamma}\dot{\delta}}{\bar{\sigma}_L}^{\dot{\delta}\alpha}{M_\alpha}^{\sigma}\sigma^{K}_{\sigma\dot{\rho}}{M_{\dot{\beta}}}^{\dot{\rho}}\varepsilon^{\dot{\gamma}\dot{\beta}} \otimes X_K = {\Lambda_L}^K \otimes X_K. \tag{32}$$

We can also multiply from the right both sides of (21) by $\bar{\sigma}_{\pm}^{N\dot{\beta}\gamma}$ and take trace over undotted indices to have (30).

To show that (29) is equal to (30), we note that

$$ \begin{aligned} \varepsilon^{\alpha\delta}\bar{\sigma}_{\pm}^{N\dot{\beta}\gamma}\varepsilon_{\gamma\delta}G_{\pm NL} &= \frac{1}{Q}\varepsilon^{\alpha\delta}\bar{\sigma}_{\pm}^{N\dot{\beta}\gamma}\varepsilon_{\gamma\delta}\varepsilon_{\dot{\nu}\dot{\mu}}{\bar{\sigma}_L}^{\dot{\mu}\rho}\sigma_{\pm N\rho\dot{\tau}}\varepsilon^{\dot{\nu}\dot{\tau}} \\ &= \varepsilon^{\alpha\delta}\delta^{\dot{\beta}}_{\dot{\tau}}\varepsilon_{\sigma\rho}\varepsilon^{\sigma\gamma}\varepsilon_{\gamma\delta}\varepsilon_{\dot{\nu}\dot{\mu}}{\bar{\sigma}_L}^{\dot{\mu}\rho}\varepsilon^{\dot{\nu}\dot{\tau}} = \varepsilon_{\dot{\nu}\dot{\mu}}{\bar{\sigma}_L}^{\dot{\mu}\alpha}\varepsilon^{\dot{\nu}\dot{\beta}}, \end{aligned} \tag{33}$$

where we have used $G_{\pm NL} = \mathrm{Tr}(\bar{\sigma}_L\sigma_{\pm N})$ and the completeness relation (27). Substituting this equality in (30), we retrieve (29).

The reality of the generators is obtained by noticing that $(\bar{\sigma}_{\pm}^{N\dot{\beta}\gamma}G_{\pm NL})^{\star} = G_{\pm LN}\bar{\sigma}_{\pm}^{N\dot{\gamma}\beta} = {\bar{\sigma}_L}^{\dot{\gamma}\beta}$ from which we get

$$(\Lambda_L{}^K)^\star = \left(\frac{1}{Q}\varepsilon^{\alpha\delta}M_\alpha{}^\sigma\sigma^K_{\sigma\dot\rho}M_{\dot\beta}{}^{\dot\rho}\bar\sigma_\pm^{N\dot\beta\gamma}\varepsilon_{\gamma\delta}G_{\pm NL}\right)^\star$$
$$= \frac{1}{Q}\varepsilon_{\dot\delta\dot\gamma}\bar\sigma_L{}^{\dot\gamma\beta}M_\beta{}^\rho\sigma^K_{\rho\dot\sigma}M_{\dot\alpha}{}^{\dot\sigma}\varepsilon^{\dot\delta\dot\alpha} = \Lambda_L{}^K$$

due to the fact that $\sigma^I_{\sigma\dot\rho}$ are hermitian. The Hopf structures of the algebra generated by $\Lambda_L{}^K$ are given by the following procedures:

1. Acting the coaction on both sides of (29), we obtain

$$\Delta(\Lambda_L{}^K) = \frac{1}{Q}\bar\sigma_{L\dot\gamma}{}^\alpha M_\alpha{}^\delta S^{-1}(M_{\dot\nu}{}^{\dot\gamma}) \otimes M_\lambda{}^\sigma\sigma_\sigma^{K\ \dot\rho}S^{-1}(M_{\dot\rho}{}^{\dot\mu})\delta^\lambda_\delta\delta^{\dot\nu}_{\dot\mu}.$$

From the completeness relation (27), we deduce

$$\Delta(\Lambda_L{}^K) = \frac{1}{Q}\bar\sigma_{L\dot\gamma}{}^\alpha M_\alpha{}^\delta\sigma_\delta^{I\ \dot\nu}S^{-1}(M_{\dot\nu}{}^{\dot\gamma}) \otimes \frac{1}{Q}\bar\sigma_{I\dot\mu}{}^\delta M_\lambda{}^\sigma\sigma_\sigma^{K\ \dot\rho}S^{-1}(M_{\dot\rho}{}^{\dot\mu})$$
$$= \Lambda_L{}^I \otimes \Lambda_I{}^K. \qquad (34)$$

2. The counity acts as

$$\varepsilon(\Lambda_L{}^K) = \frac{1}{Q}\bar\sigma_{L\dot\gamma}{}^\alpha\varepsilon(M_\alpha{}^\sigma)\sigma_\sigma^{K\ \dot\rho}\varepsilon(S^{-1}(M_{\dot\rho}{}^{\dot\gamma}))$$
$$= \frac{1}{Q}\bar\sigma_{L\dot\gamma}{}^\alpha\delta^\sigma_\alpha\sigma_\sigma^{K\,\dot\rho}\delta^{\dot\gamma}_{\dot\rho} = \frac{1}{Q}\bar\sigma_{L\dot\gamma}{}^\alpha\sigma_\alpha{}^{K\dot\gamma} = \frac{1}{Q}G_{\pm LN}\mathrm{Tr}(\bar\sigma^N_\pm\sigma^K)$$
$$= G_{\pm LN}G_\pm^{NK} = \delta^K_L.$$

3. Finally, applying the antipode on both sides of (29), we get

$$S(\Lambda_L{}^K) = \frac{1}{Q}\varepsilon^{\dot\gamma\dot\beta}S(M_{\dot\beta}{}^{\dot\rho})\sigma^K_{\sigma\dot\rho}S(M_\alpha{}^\sigma)\bar\sigma_L^{\dot\delta\alpha}\varepsilon_{\dot\gamma\dot\delta}$$
$$= \frac{1}{Q}M_{\dot\nu}{}^{\dot\gamma}\varepsilon^{\dot\nu\dot\rho}\sigma^K_{\sigma\dot\rho}\varepsilon_{\alpha\lambda}M_\mu{}^\lambda\varepsilon^{\mu\sigma}\bar\sigma_L{}^{\dot\delta\alpha}\varepsilon_{\dot\gamma\dot\delta}.$$

It now follows from $M_{\dot\nu}{}^{\dot\gamma}M_\mu{}^\lambda = R^{\mp\tau\dot\xi}_{\dot\nu\mu}M_\tau{}^\delta M_{\dot\xi}{}^{\dot\rho}R^{\pm\dot\gamma\lambda}_{\delta\dot\rho}$, obtained by multiplying from the right both sides of (18) by $R^{\mp\alpha\dot\sigma}_{\tau\nu}$ and by using (19), that

$$S(\Lambda_L{}^K) = \frac{1}{Q}R^{\mp\tau\dot\xi}_{\dot\nu\mu}M_\tau{}^\delta M_{\dot\xi}{}^{\dot\beta}R^{\pm\dot\gamma\lambda}_{\delta\dot\beta}\varepsilon^{\dot\nu\dot\rho}\sigma^K_{\sigma\dot\rho}\varepsilon_{\alpha\lambda}\varepsilon^{\mu\sigma}\bar\sigma_L{}^{\dot\delta\alpha}\varepsilon_{\dot\gamma\dot\delta}$$
$$= \frac{1}{Q}R^{\mp\tau\dot\xi}_{\dot\nu\mu}M_\tau{}^\delta M_{\dot\xi}{}^{\dot\beta}\varepsilon^{\dot\nu\dot\rho}\sigma^K_{\sigma\dot\rho}\varepsilon^{\mu\sigma}\varepsilon_{\dot\gamma\dot\delta}R^{\pm\dot\gamma\lambda}_{\delta\dot\beta}\varepsilon_{\alpha\lambda}\bar\sigma_L{}^{\dot\delta\alpha}$$
$$= \frac{1}{Q}\varepsilon^{\mu\sigma}\sigma^K_{\sigma\dot\rho}\varepsilon^{\dot\nu\dot\rho}R^{\mp\tau\dot\xi}_{\dot\nu\dot\mu}M_\tau{}^\delta M_{\dot\xi}{}^{\dot\beta}\sigma_{\pm L\delta\dot\beta} = \frac{1}{Q}R^{\mp\tau\dot\xi}_{\dot\nu\mu}\varepsilon^{\mu\sigma}\sigma_\sigma^{K\,\dot\nu}M_\tau{}^\delta M_{\dot\xi}{}^{\dot\beta}\sigma_{\pm L\delta\dot\beta}, \qquad (35)$$

where we have used (23) and $G_{\pm LI}\sigma^I_{\alpha\dot\beta} = \sigma_{\pm L\alpha\dot\beta}$ to pass from the second line to the third. On the other hand, we have

$$\bar\sigma_N{}^{\dot\alpha\beta}G_\pm^{NK} = \frac{1}{Q}\bar\sigma_N{}^{\dot\alpha\beta}\varepsilon_{\dot\nu\dot\gamma}\bar\sigma_\pm^{N\dot\gamma\delta}\sigma^K_{\delta\dot\tau}\varepsilon^{\dot\nu\dot\tau} = \frac{1}{Q}\bar\sigma_N{}^{\dot\alpha\beta}\varepsilon_{\dot\nu\dot\gamma}\varepsilon^{\dot\gamma\dot\lambda}R^{\mp\tau\dot\rho}_{\dot\lambda\sigma}\varepsilon^{\sigma\delta}\sigma^N_{\tau\dot\rho}\sigma^K_{\delta\dot\tau}\varepsilon^{\dot\nu\dot\tau},$$

where we have used (24) and (22). Using the completeness relation (26), we obtain

$$\bar{\sigma}_N{}^{\dot{\alpha}\beta} G_{\pm}^{NK} = \delta_\tau^\beta \varepsilon_{\dot{\rho}\dot{\mu}} \varepsilon^{\dot{\alpha}\dot{\mu}} R_{\dot{\nu}\sigma}^{\mp\tau\dot{\rho}} \varepsilon^{\sigma\delta} \sigma_{\delta\dot{\tau}}^K \varepsilon^{\dot{\nu}\dot{\tau}}$$

or

$$\bar{\sigma}_{N\dot{\alpha}}{}^{\beta} G_{\pm}^{NK} = \varepsilon_{\dot{\rho}\dot{\alpha}} R_{\dot{\nu}\sigma}^{\mp\beta\dot{\rho}} \varepsilon^{\sigma\delta} \sigma_\delta^{K\ \dot{\nu}}. \tag{36}$$

Substituting this equality in (35), we get

$$\begin{aligned} S(\Lambda_L{}^K) &= \frac{1}{Q} \bar{\sigma}_{N\dot{\alpha}}{}^{\tau} M_\tau{}^\delta \sigma_{L\delta\dot{\beta}} M_{\dot{\rho}}{}^{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\rho}} G_{\pm}{}^{NK} \\ &= \frac{1}{Q} G_{\pm LM} \left(\bar{\sigma}_{N\dot{\alpha}}{}^{\tau} M_\tau{}^\delta \sigma_{\delta\dot{\beta}}^M M_{\dot{\rho}}{}^{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\rho}} \right) G_{\pm}^{NK} = G_{\pm LM} \Lambda_N{}^M G_{\pm}^{NK} \end{aligned}$$

from which we deduce the orthogonality conditions (31) which may also be checked directly by replacing the generators $\Lambda_L{}^K$ by their expression (29). Then $\Lambda_L{}^K$ generate a Lorentz algebra $\mathcal{L} \subset \mathcal{A}$ (Hopf algebra whose generators are subject to Lorentz group conditions (31)). □

The noncommutativity between the generators $\Lambda_L{}^K$ and the elements of $\mathcal{A}$ are given by the following theorem.

Theorem 3.2. *There exist functionals $F_{\pm L}{}^K : \mathcal{A} \to C$ given by*

$$F_{\pm L}{}^K = \frac{1}{Q} (\tilde{f}_{\mp\dot{\beta}}{}^{\dot{\alpha}} \bar{\sigma}_{L\dot{\alpha}}{}^{\delta} \star f_{\pm\delta}{}^{\gamma} \sigma_\gamma^{K\ \dot{\beta}}) \tag{37}$$

satisfying the following:

(a)

$$\Lambda_L{}^I (F_{\pm I}{}^K \star a) = (a \star F_{\pm L}{}^I) \Lambda_I{}^K \tag{38}$$

(b)

$$(F_{\pm L}{}^K (a))^\star = F_{\pm L}{}^K (S(a^\star)) \tag{39}$$

(c)

$$F_{\pm L}{}^K (ab) = F_{\pm L}{}^I (a) F_{\pm I}{}^K (b) \tag{40}$$

$$F_{\pm L}{}^K (\varepsilon(a)) = \delta_L^K \varepsilon(a) \tag{41}$$

$$(F_{\pm L}{}^I \star F_{\pm I}{}^K \circ S)(a) = (F_{\pm L}{}^I \circ S \star F_{\pm I}{}^K)(a) = \delta_L{}^K \varepsilon(a) \tag{42}$$

(d)

$$\mathcal{R}_{KL}^{\pm NM} G_{\pm}{}^{KL} = G_{\pm}{}^{NM}, \qquad \mathcal{R}_{KL}^{\pm NM} G_{\pm NM} = G_{\pm KL}, \tag{43}$$

where $\mathcal{R}_{KL}^{\pm NM} = F_{\pm K}{}^M (\Lambda_L{}^N)$ satisfy the Yang–Baxter equations and the Hecke relations

$$(\mathcal{R}^\pm + a^{\pm 2})(\mathcal{R}^\pm + a^{\mp 2})(\mathcal{R}^\pm - 1) = 0. \tag{44}$$

Proof. (a) $X_{\pm\alpha\dot\beta}$ are right invariant bases of $\mathcal{A}$–$\mathcal{M}_\pm$ bimodules transforming under the $\mathcal{L}$ algebra as (21). Then we can follow the Woronowicz formalism of the bicovariant bimodule [4] to state

$$\begin{aligned} X_{\pm\alpha\dot\beta}a &= (a\star(f_{\mp\dot\beta}{}^{\dot\rho}\star f_{\pm\alpha}{}^{\sigma}))X_{\pm\sigma\dot\rho} = f_{\mp\dot\beta}{}^{\dot\rho}(a_{(1)})f_{\pm\alpha}{}^{\sigma}(a_{(2)})a_{(3)}X_{\pm\sigma\dot\rho},\\ aX_{\pm\alpha\dot\beta} &= X_{\pm\sigma\dot\rho}(a\star(f_{\mp\dot\beta}{}^{\dot\rho}\star f_{\pm\alpha}{}^{\sigma})\circ S) = X_{\pm\sigma\dot\rho}f_{\pm\alpha}{}^{\sigma}(S(a_{(1)}))f_{\mp\dot\beta}{}^{\dot\rho}(S(a_{(2)}))a_{(3)} \end{aligned} \tag{45}$$

for any $a\in\mathcal{A}$. The convolution product of two functionals is defined as $(f_1\star f_2)(a) = (f_1\otimes f_2)\Delta(a)$. This choice of functionals is justified by the fact that if we apply the $\star$ involution on both sides of the first relation of (45) and we use the second relation with $a=a^\star$, we obtain

$$\begin{aligned} a^\star X_{\pm\beta\dot\alpha} &= X_{\pm\rho\dot\sigma}(f_{\mp\dot\beta}{}^{\dot\rho}(a_{(1)}))^\star(f_{\pm\alpha}{}^{\sigma}(a_{(2)}))^\star(a_{(3)})^\star\\ &= X_{\pm\rho\dot\sigma}f_{\pm\beta}{}^{\rho}(S(a^\star_{(1)}))f_{\mp\dot\alpha}{}^{\dot\sigma}(S(a^\star_{(2)}))(a_{(3)})^\star \end{aligned} \tag{46}$$

which is consistent with (14). Now, the bicovariant bimodule formalism [4] can also be applied to the $\mathcal{A}$–$\mathcal{M}_\pm$ bimodules to get

$$M_\alpha{}^{\rho}M_{\dot\beta}{}^{\dot\sigma}((f_{\mp\dot\sigma}{}^{\dot\delta}\star f_{\pm\rho}{}^{\gamma})\star a) = (a\star(f_{\mp\dot\beta}{}^{\dot\sigma}\star f_{\pm\alpha}{}^{\rho}))M_\rho{}^{\gamma}M_{\dot\sigma}{}^{\dot\delta}$$

from which we deduce

$$M_\alpha{}^{\sigma}S^{-1}(M_{\dot\mu}{}^{\dot\nu})(\tilde f_{\mp\dot\beta}{}^{\dot\mu}\star f_{\pm\sigma}{}^{\gamma}\star a) = (a\star(\tilde f_{\mp\dot\mu}{}^{\dot\mu}\star f_{\pm\alpha}{}^{\sigma})M_\sigma{}^{\gamma}S^{-1}(M_{\dot\beta}{}^{\dot\mu}),$$

where we have used (15). Note that this equation is a combination of the first equation of (9) and the second equation of (10).

Multiplying now both sides from the left by $\bar\sigma_{L\dot\nu}{}^{\alpha}$ and from right by $\sigma_\gamma^{K\,\dot\beta}$ and using the completeness relation (27), we obtain

$$\begin{aligned} &\frac{1}{Q}\bar\sigma_{L\dot\nu}{}^{\alpha}M_\alpha{}^{\sigma}\sigma_\sigma^{I\,\dot\mu}S^{-1}(M_{\dot\mu}{}^{\dot\nu})((\tilde f_{\mp\dot\beta}{}^{\dot\tau}\bar\sigma_{I\dot\tau}{}^{\delta}\star f_{\pm\delta}{}^{\gamma}\sigma_\gamma^{K\,\dot\beta})\star a)\\ &= (a\star(\tilde f_{\mp\dot\mu}{}^{\dot\nu}\bar\sigma_{L\dot\nu}{}^{\alpha}\star f_{\pm\alpha}{}^{\sigma}\sigma_\sigma^{I\,\dot\mu}))\frac{1}{Q}\bar\sigma_{I\dot\tau}{}^{\delta}M_\delta{}^{\gamma}\sigma_\gamma^{K\,\dot\beta}S^{-1}(M_{\dot\beta}{}^{\dot\tau}) \end{aligned}$$

yielding

$$\Lambda_L{}^{I}(F_{\pm I}{}^{K}\star a) = (a\star F_{\pm L}{}^{I})\Lambda_I{}^{K}\quad\text{with}\quad F_{\pm L}{}^{K} = \frac{1}{Q}(\tilde f_{\mp\dot\beta}{}^{\dot\alpha}\bar\sigma_{L\dot\alpha}{}^{\delta}\star f_{\pm\delta}{}^{\gamma}\sigma_\gamma^{K\,\dot\beta}).$$

(b) Using (14),(15) and (33), we get

$$
\begin{aligned}
(F_{\pm L}{}^{K}(a))^{\star} &= \frac{1}{Q}(G_{\pm LN}\varepsilon_{\dot\alpha\dot\rho}\tilde f_{\mp\dot\beta}{}^{\dot\alpha}(a_{(1)})\bar\sigma_{\pm}{}^{N\dot\rho\delta} f_{\pm\delta}{}^{\gamma}(a_{(2)})\sigma^{K}_{\gamma\dot\lambda}\varepsilon^{\dot\beta\dot\lambda})^{\star}\\
&= \frac{1}{Q}\varepsilon^{\lambda\beta}(f_{\pm\delta}{}^{\gamma}(a_{(2)}))^{\star}\sigma^{K}_{\lambda\dot\gamma}(\tilde f_{\pm\dot\beta}{}^{\dot\alpha}(a_{(1)}))^{\star}\bar\sigma_{\pm}^{N\dot\delta\rho}G_{\pm NL}\varepsilon_{\rho\alpha}\\
&= \frac{1}{Q}\varepsilon^{\lambda\beta} f_{\mp\dot\delta}{}^{\dot\gamma}(S(a^{\star}_{(2)}))\sigma^{K}_{\lambda\dot\gamma}\tilde f_{\pm\beta}{}^{\alpha}(S(a^{\star}_{(1)}))\varepsilon_{\dot\nu\dot\mu}\bar\sigma_{L}{}^{\dot\mu\rho}\varepsilon_{\alpha\rho}\varepsilon^{\dot\nu\dot\delta}\\
&= \frac{1}{Q}\varepsilon^{\dot\nu\dot\delta} f_{\mp\dot\delta}{}^{\dot\gamma}(S(a^{\star}_{(2)}))\varepsilon_{\dot\gamma\dot\xi}\sigma_{\lambda}^{K\ \dot\xi}\varepsilon^{\lambda\beta}\tilde f_{\pm\beta}{}^{\alpha}(S(a^{\star}_{(1)}))\varepsilon_{\alpha\rho}\bar\sigma_{L\dot\nu}{}^{\rho}\\
&= \frac{1}{Q}\tilde f_{\mp\dot\xi}{}^{\dot\nu}(S(a^{\star}_{(2)}))\bar\sigma_{L\dot\nu}{}^{\rho} f_{\pm\rho}{}^{\lambda}(S(a^{\star}_{(1)}))\sigma_{\lambda}^{K\ \dot\xi} = F_{\pm L}{}^{K}(S(a^{\star})).
\end{aligned}
$$

(c) Eq. (40) is deduced directly from (6) and (7) and the completeness relation (27) as

$$
\begin{aligned}
F_{\pm L}{}^{K}(ab) &= \frac{1}{Q}\tilde f_{\mp\dot\beta}{}^{\dot\alpha}(a_{(1)}b_{(1)})\bar\sigma_{L\dot\alpha}{}^{\delta} f_{\pm\delta}{}^{\gamma}(a_{(2)}b_{(2)})\sigma_{\gamma}^{K\ \dot\beta}\\
&= \frac{1}{Q}\tilde f_{\mp\dot\nu}{}^{\dot\alpha}(a_{(1)})\tilde f_{\mp\dot\beta}{}^{\dot\nu}(b_{(1)})\bar\sigma_{L\dot\alpha}{}^{\delta} f_{\pm\delta}{}^{\mu}(a_{(2)}) f_{\pm\mu}{}^{\gamma}(b_{(2)})\sigma_{\gamma}^{K\ \dot\beta}\\
&= \frac{1}{Q}\tilde f_{\mp\dot\rho}{}^{\dot\alpha}(a_{(1)})\tilde f_{\mp\dot\beta}{}^{\dot\nu}(b_{(1)})\bar\sigma_{L\dot\alpha}{}^{\delta} f_{\pm\delta}{}^{\mu}(a_{(2)}) f_{\pm\tau}{}^{\gamma}(b_{(2)})\sigma_{\gamma}^{K\ \dot\beta}\delta^{\tau}_{\mu}\delta^{\dot\rho}_{\dot\nu}\\
&= \frac{1}{Q}(\tilde f_{\mp\dot\rho}{}^{\dot\alpha}\bar\sigma_{L\dot\alpha}{}^{\delta}\star f_{\pm\delta}{}^{\mu}\sigma_{\mu}^{I\ \dot\rho})(a)\frac{1}{Q}(\tilde f_{\mp\dot\beta}{}^{\dot\nu}\bar\sigma_{I\dot\nu}{}^{\tau}\star f_{\pm\tau}{}^{\gamma}\sigma_{\gamma}^{K\ \dot\beta})(b)\\
&= F_{\pm L}{}^{I}(a)F_{\pm I}{}^{I}(b).
\end{aligned}
$$

We also have

$$
\begin{aligned}
F_{\pm L}{}^{K}(\varepsilon(a)) &= \frac{1}{Q}(\tilde f_{\mp\dot\beta}{}^{\dot\alpha}(\varepsilon(a_{(1)}))\bar\sigma_{L\dot\alpha}{}^{\delta} f_{\pm\delta}{}^{\gamma}(\varepsilon(a_{(2)}))\sigma_{\gamma}^{K\ \dot\beta} = \frac{1}{Q}\bar\sigma_{L\dot\alpha}{}^{\delta}\sigma_{\delta}^{K\ \dot\alpha}\varepsilon(a)\\
&= \delta_{L}{}^{K}\varepsilon(a),
\end{aligned}
$$

where we have used (8). Eq. (42) can be deduced directly from (40) and (41).

(d) Applying (37) on (29), we get

$$
\begin{aligned}
\mathcal{R}^{\pm NM}_{LK}G^{LK}_{\pm} &= \frac{1}{Q}(\tilde f_{\mp\dot\beta}{}^{\dot\tau}\bar\sigma_{L\dot\tau}{}^{\delta}\star f_{\pm\delta}{}^{\gamma}\sigma_{\gamma}^{M\ \dot\beta})\left(\frac{1}{Q}\bar\sigma_{K\dot\nu}{}^{\sigma}M_{\sigma}{}^{\rho}\sigma_{\rho}^{N\ \dot\lambda}S^{-1}(M_{\dot\lambda}{}^{\dot\nu})\right)G_{\pm}{}^{LK}\\
&= \frac{1}{Q^{2}}\tilde f_{\mp\dot\beta}{}^{\dot\tau}(M_{\sigma}{}^{\alpha}S^{-1}(M_{\dot\mu}{}^{\dot\nu}))\bar\sigma_{L\dot\tau}{}^{\delta}\bar\sigma_{K\dot\nu}{}^{\sigma} f_{\pm\delta}{}^{\gamma}(M_{\alpha}{}^{\rho}S^{-1}(M_{\dot\lambda}{}^{\dot\mu}))\\
&\quad\times\sigma_{\gamma}^{M\ \dot\beta}\sigma_{\rho}^{N\ \dot\lambda}G_{\pm}{}^{LK}.
\end{aligned}
$$

Using $G_{\pm}{}^{LK}\bar\sigma_{L\dot\tau}{}^{\delta}\bar\sigma_{K\dot\nu}{}^{\sigma} = \bar\sigma_{L\dot\tau}{}^{\delta}\bar\sigma_{\pm\ \dot\nu}^{L\ \sigma} = R^{\pm\alpha\dot\beta}_{\dot\nu\mu}\varepsilon^{\mu\sigma}\sigma^{L}_{\alpha\dot\beta}\bar\sigma_{L\dot\tau}{}^{\delta} = -Q(a^{\pm(1/2)}\varepsilon^{\delta\nu}\varepsilon^{\tau\sigma} + a^{\mp(1/2)}\varepsilon^{\delta\tau}\varepsilon^{\sigma\nu})$, obtained from (22), the completeness relation (16), (27) and the form of the R matrices, we get

$$
\begin{aligned}
&-\frac{1}{Q}\tilde f_{\mp\dot\kappa}{}^{\dot\tau}(M_{\sigma}{}^{\alpha})\tilde f_{\mp\dot\beta}{}^{\dot\kappa}(S^{-1}(M_{\dot\mu}{}^{\dot\nu})) f_{\pm\delta}{}^{\xi}(M_{\alpha}{}^{\rho}) f_{\pm\xi}{}^{\gamma}(S^{-1}(M_{\dot\lambda}{}^{\dot\mu}))(a^{\pm(1/2)}\varepsilon^{\delta\nu}\varepsilon^{\tau\sigma}\\
&+a^{\mp(1/2)}\varepsilon^{\delta\tau}\varepsilon^{\sigma\nu}) = -\frac{1}{Q}R^{\mp\alpha\kappa}_{\tau\sigma}R^{\mp\mu\beta}_{\kappa\nu}R^{\pm\rho\xi}_{\delta\alpha}R^{\pm\lambda\gamma}_{\xi\mu}(a^{\pm(1/2)}\varepsilon^{\delta\nu}\varepsilon^{\tau\sigma} + a^{\mp(1/2)}\varepsilon^{\delta\tau}\varepsilon^{\sigma\nu}),
\end{aligned}
$$

where we have used Eq. (17). An explicit computation gives

$$R^{\mp\alpha\kappa}_{\tau\sigma} R^{\mp\mu\beta}_{\kappa\nu} R^{\pm\rho\xi}_{\delta\alpha} R^{\pm\lambda\gamma}_{\xi\mu} \varepsilon^{\delta\nu}\varepsilon^{\tau\sigma} = \varepsilon^{\rho\beta}\varepsilon^{\lambda\gamma} \quad \text{and} \quad R^{\mp\alpha\kappa}_{\tau\sigma} R^{\mp\mu\beta}_{\kappa\nu} R^{\pm\rho\xi}_{\delta\alpha} R^{\pm\lambda\gamma}_{\xi\mu} \varepsilon^{\delta\tau}\varepsilon^{\sigma\nu} = \varepsilon^{\rho\lambda}\varepsilon^{\gamma\beta}$$

from which we deduce

$$\begin{aligned} \mathcal{R}^{\pm NM}_{LK} G^{\pm LK} &= -\frac{1}{Q}(a^{\pm(1/2)}\varepsilon^{\rho\beta}\varepsilon^{\lambda\gamma} + a^{\mp(1/2)}\varepsilon^{\rho\lambda}\varepsilon^{\gamma\beta})\sigma^{M\,\dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho} \\ &= \frac{1}{Q}(a^{\pm(1/2)}\varepsilon^{\rho\beta}\varepsilon_{\dot\gamma\dot\lambda} - a^{\mp(1/2)}\varepsilon^{\rho\lambda}\varepsilon^{\gamma\beta})\sigma^{M\,\dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho} \\ &= \frac{1}{Q}(a^{\pm(1/2)}\sigma^{N\beta}_{\dot\gamma}\sigma^{M\,\dot\beta}_{\gamma} - a^{\mp(1/2)}\sigma^{N\lambda\dot\lambda}\sigma^{M\beta\dot\beta}) = G^{NM}_{\pm}, \end{aligned}$$

where we have used Eqs. (16) and (25). A similar calculation gives

$$\mathcal{R}^{\pm NM}_{LK} G_{\pm NM} = \frac{1}{Q}(a^{\mp(1/2)}\bar\sigma_{L\,\delta}{}^{\dot\beta}\bar\sigma_{K\dot\delta}{}^{\beta} - a^{\pm(1/2)}\bar\sigma_{L\dot\beta\beta}\bar\sigma_{K\dot\delta\delta}) = G_{\pm LK}.$$

The Yang–Baxter equations may be obtained by applying ${F_{\pm L}}^K$ on both sides of (38) for $a = {\Lambda_N}^M$ and then using (40). Using the Hecke relations of the R matrices, we obtain after an explicit straightforward calculation

$$\begin{aligned} \mathcal{R}^{\pm NM}_{IJ}\mathcal{R}^{\pm IJ}_{LK} &= (2 - a^{\pm2} - a^{\mp2})\mathcal{R}^{\pm NM}_{LK} + \delta^N_L\delta^M_K \\ &\quad + \frac{1}{Q^2}a^{\mp2}(1 - a^{\pm2})R^{\pm\rho\xi}_{\delta\alpha}R^{\pm\lambda\gamma}_{\xi\kappa}R^{\mp\alpha\kappa}_{\tau\sigma}\sigma^{M\,\dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho}\bar\sigma_{L\dot\tau}{}^{\delta}\bar\sigma_{K\dot\beta}{}^{\sigma} \\ &\quad + \frac{1}{Q^2}a^{\pm2}(1 - a^{\mp2})R^{\mp\mu\beta}_{\xi\nu}R^{\pm\lambda\gamma}_{\alpha\mu}R^{\mp\alpha\xi}_{\tau\sigma}\sigma^{M\ \dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho}\bar\sigma_{L\dot\tau}{}^{\rho}\bar\sigma_{K\dot\nu}{}^{\sigma}, \\ &\quad \frac{1}{Q^2}R^{\pm\rho\xi}_{\delta\alpha}R^{\pm\lambda\gamma}_{\xi\kappa}R^{\pm\alpha\kappa}_{\tau\sigma}\sigma^{M\,\dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho}\bar\sigma_{I\dot\tau}{}^{\delta}\bar\sigma_{J\dot\beta}{}^{\sigma}\mathcal{R}^{\pm IJ}_{LK} \\ &= (1 - a^{\pm2})\mathcal{R}^{\pm NM}_{LK} + \frac{a^{\pm2}}{Q^2}R^{\mp\mu\beta}_{\xi\nu}R^{\pm\lambda\gamma}_{\kappa\mu}R^{\mp\kappa\xi}_{\tau\sigma}\sigma^{M\ \dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho}\bar\sigma_{L\dot\tau}{}^{\rho}\bar\sigma_{K\dot\nu}{}^{\sigma}, \\ &\quad \frac{1}{Q^2}R^{\mp\mu\beta}_{\xi\nu}R^{\pm\lambda\gamma}_{\alpha\mu}R^{\mp\alpha\xi}_{\tau\sigma}\sigma^{M\ \dot\beta}_{\gamma}\sigma^{N\,\dot\lambda}_{\rho}\bar\sigma_{I\dot\tau}{}^{\rho}\bar\sigma_{J\dot\nu}{}^{\sigma}\mathcal{R}^{\pm IJ}_{LK} \\ &= (1 - a^{\mp2})\mathcal{R}^{\pm NM}_{LK} + \frac{a^{\mp2}}{Q^2}R^{\pm\delta\alpha}_{\xi\kappa}R^{\pm\lambda\gamma}_{\alpha\mu}R^{\mp\kappa\mu}_{\tau\sigma}\sigma^{M\ \dot\beta}_{\gamma}\sigma^{N\ \dot\delta}_{\rho}\bar\sigma_{L\dot\tau}{}^{\xi}\bar\sigma_{K\dot\beta}{}^{\sigma} \end{aligned}$$

leading to (44). □

As for the $\mathcal{A}$–Γ bimodule, the existence of two functionals ${F_{\pm L}}^K$ leads to right invariant bases $X_{\pm I}$ of the $\mathcal{A}$–$\mathcal{M}_\pm$ bimodules satisfying

$$\begin{aligned} X_{\pm L}a &= (a \star {F_{\pm L}}^K)X_{\pm K}, \quad X^{(\pm)L}_{\pm}a = (a \star \tilde F^{(\pm)\ \ L}_{\pm K})X^{(\pm)K}_{\pm} \\ aX_{\pm L} &= X_{\pm K}(a \star {F_{\pm L}}^K \circ S), \quad aX^{(\pm)L}_{\pm} = X^{(\pm)K}_{\pm}(a \star \tilde F^{(\pm)\ \ L}_{\pm K} \circ S) \end{aligned} \tag{47}$$

for any $a \in \mathcal{A}$. The indices are raised and lowered by using the Minkowski metric as $X_L G^{LK}_{\pm} = X^{(\pm)K}$. Following the same formalism applied to the $\mathcal{A}$–Γ bimodule, we may show that

$$F_{\pm L}{}^{K} = G_{\pm}^{KM}\tilde{F}_{\pm M}^{(\pm)}{}^{N} G_{\pm NL} \quad \text{and} \quad \tilde{F}_{\pm L}^{(\pm)}{}^{K} \circ S = F_{\pm L}{}^{K}$$

implying

$$G_{\pm L}{}^{MN} F_{\pm N}{}^{L} \star F_{\pm M}{}^{K}(a) = G_{\pm}{}^{KL}\varepsilon(a) \quad \text{and}$$
$$G_{\pm KL} F_{\pm N}{}^{L} \star F_{\pm M}{}^{K}(a) = G_{\pm MN}\varepsilon(a) \tag{48}$$

for any $a \in \mathcal{A}$.

4. Quantum Minkowski space

We consider the elements $X_{\pm I}$ of right invariant bases of the $\mathcal{A}$–$\mathcal{M}_{\pm}$ bimodules as coordinates which span the Minkowski vector spaces $\mathcal{M}_{\pm}$ over the field R. The Minkowski space $\mathcal{M}_{+}(\mathcal{M}_{-})$ is equipped with coordinates $X_{+I}(X_{-})$ and the metric $G_{+IJ}(G_{-IJ})$. Eq. (32) shows that $\Delta_L : \mathcal{M} \to \mathcal{L} \otimes \mathcal{M}$ is a corepresentation of $\mathcal{L}$ in the vector space $\mathcal{M}$. In fact, from (32) and (34) and the action of the counity on the generators of $\mathcal{L}$, it is easy to verify $(id \otimes \Delta_L)\Delta_L = (\Delta \otimes id)\Delta_L$ and $(\varepsilon \otimes id)\Delta_L = id$.

In the following we assume that the coordinates $X_{\pm I}$ commute, in the quantum sense, with themselves and with the elements (spinors) of the right invariant basis of $\mathcal{A}$–Γ bimodule. To carry this quantum symmetrization, we consider the bicovariant bimodule automorphism σ [4] such that for any $a, b \in \mathcal{A}$ and the left invariant element $\tilde{X}_{\pm I} = S(\Lambda_I{}^J)X_{\pm J}$, $\tilde{\theta}_{\pm\alpha} = S(M_\alpha{}^\beta)\theta_{\pm\beta}$ or $\tilde{\theta}_{\pm\dot\alpha} = S(M_{\dot\alpha}{}^{\dot\beta})\theta_{\pm\dot\beta}$, we have

$$\begin{aligned}\sigma(\tilde{X}_{(a)L} \otimes X_{(b)K} &= X_{(b)K} \otimes \tilde{X}_{(a)L} = X_{(b)K} \otimes S(\Lambda_L{}^N)X_{(a)N}\\ &= S(\Lambda_L{}^N)\sigma(X_{(a)N} \otimes X_{(b)K})\\ &= F_{(b)K}{}^{Q}(S(\Lambda_P{}^N))S(\Lambda_L{}^P)(X_{(b)Q} \otimes X_{(a)N}),\end{aligned}$$

where $a, b = \pm$ or $\mp$. From the latter equation we deduce

$$\sigma(X_{(a)L} \otimes X_{(b)K}) = F_{(b)K}{}^{N}(S(\Lambda_L{}^M))(X_{(b)N} \otimes X_{(a)M}).$$

For the spinors, it suffices to replace $\Lambda_L{}^K$ by $M_\alpha{}^\beta$ or $M_{\dot\alpha}{}^{\dot\beta}$ and $X_{(a)I}$ by $\theta_{(a)\alpha}$ or $\theta_{(a)\dot\alpha}$. The symmetrization of the product is defined as

$$\begin{aligned}X_{(a)L}X_{(b)K} &= F_{(b)K}{}^{N}(S(\Lambda_L{}^M))(X_{(b)N} \otimes X_{(a)M}),\\ \theta_{(a)\alpha}X_{(b)K} &= F_{(b)K}{}^{N}(S(M_\alpha{}^\beta))(X_{(b)N}\theta_{(a)\beta}) \quad \text{or}\\ \theta_{(a)\dot\alpha}X_{(b)K} &= F_{(b)K}{}^{N}(S(M_{\dot\alpha}{}^{\dot\beta}))(X_{(b)N}\theta_{(a)\dot\beta}).\end{aligned} \tag{49}$$

From this, we state

Theorem 4.1. *The lengths, $G_{\pm}{}^{IJ}X_{\pm I}X_{\pm J}$ are bi-invariant and central.*

Proof. By construction, $X_{\pm I}$ are right invariant. As a consequence of the orthogonality condition of the generators of $\mathcal{L}$, we may easily see from the transformations of $X_{\pm I}$ (34) that $G_{\pm}{}^{IJ}X_{\pm I}X_{\pm J}$ is left invariant. From (47) and (48), we get, for any $a \in \mathcal{A}$,

$$G_{\pm}{}^{IJ}X_{\pm I}X_{\pm J}a = (a\star G_{\pm}{}^{IJ}(F_{\pm J}{}^{L}\star F_{\pm I}{}^{K}))X_{\pm K}X_{\pm L} = (a\star\varepsilon)G_{\pm}{}^{KL}X_{\pm K}X_{\pm L} = aG_{\pm}{}^{KL}X_{\pm K}X_{\pm L} \quad (50)$$

which shows that the norm commutes with any $a \in \mathcal{A}$. From (48) and (49), we obtain

$$\begin{aligned} X_{(a)P}G_{\pm}{}^{IJ}X_{\pm I}X_{\pm J} &= G_{\pm}{}^{IJ}F_{\pm I}{}^{N}(S(\Lambda_P{}^{M}))F_{\pm J}{}^{K}(S(\Lambda_M{}^{L}))X_{\pm N}X_{\pm K}X_{(a)L} \\ &= G_{\pm}{}^{IJ}(F_{\pm J}{}^{K}\star F_{\pm I}{}^{N})(S(\Lambda_P{}^{L}))X_{\pm N}X_{\pm K}X_{(a)L} \\ &= G_{\pm}{}^{NK}X_{\pm N}X_{\pm K}X_{(a)P}. \end{aligned} \quad (51)$$

The same results may be obtained by replacing $X_{(a)P}$ by $\theta_{(a)\alpha}$ or $\theta_{(a)\dot\alpha}$ which show that the length also commutes with the quantum coordinates and the spinors. □

Remark 4.1.

- $G_{\pm}{}^{LK}X_{\pm L}X_{\pm K}$ *is bi-invariant and real. Since it commutes with everything, it is of the form* $\lambda I_{\mathcal{L}}$ *with* λ *a real number*.
- *In the classical limit* $F_{\pm L}{}^{N}(\Lambda_K{}^{M}) = \delta_L^N\delta_K^M$, *therefore, both quantum Minkowksi spaces* $\mathcal{M}_{\pm}$ *reduce to the same classical Minkowski space*.
- *The quantitites* $G_{\pm}{}^{IJ}X_{\mp I}X_{\mp J}$ *and* $G_{\pm}{}^{IJ}X_{\mp I}X_{\pm J}$ *are bi-invariant but not central*.

5. Concluding remarks

In this paper we have showed how all properties of the quantum Lorentz group and Minkowski space can be derived from those of the quantum $SL(2,C)$ group and its spinorial representations. The Minkowski space $\mathcal{M}_{+}$ can be equipped by quantum real coordinates $X_{+I}(X_{-I})$ and a hermitian metric $G_{+}{}^{IJ}(G_{-}{}^{IJ})$ giving a length $G_{+}{}^{IJ}X_{+I}X_{+J}(G_{-}{}^{IJ}X_{-I}X_{-J})$ central and bi-invariant under quantum Lorentz and $SL(2,C)$ groups. The commutation relations are given by

$$\Lambda_L{}^{I}\Lambda_N{}^{P}\mathcal{R}^{\pm MK}_{IP} = \mathcal{R}^{\pm IP}_{LN}\Lambda_I{}^{M}\Lambda_P{}^{K}, \quad (52)$$

$$X_{\pm L}\Lambda_N{}^{M} = \mathcal{R}^{\pm PI}_{LN}\Lambda_P{}^{M}X_{\pm I}, \quad (53)$$

$$X_{\pm N}X_{\pm M} = \mathcal{R}^{\pm LK}_{NM}X_{\pm L}X_{\pm K}, \quad (54)$$

where the matrices $\mathcal{R}^{\pm NM}_{LK} = F_{\pm L}{}^{M}(\Lambda_K{}^{N})$ are given by

$$\mathcal{R}^{\pm NM}_{LK} = \frac{1}{Q^2}R^{\mp\alpha\kappa}_{\tau\sigma}R^{\mp\mu\beta}_{\kappa\nu}R^{\pm\rho\varepsilon}_{\delta\alpha}R^{\pm\lambda\gamma}_{\varepsilon\mu}\sigma^{M}_{\gamma}{}^{\dot\beta}\sigma^{N}_{\rho}{}^{\dot\lambda}\bar\sigma_{L\dot\tau}{}^{\delta}\bar\sigma_{K\dot\nu}{}^{\sigma}. \quad (55)$$

By writing the R matrix in terms of projectors $\mathcal{P}_{\mathcal{A}\rho\delta}{}^{\alpha\beta} = -(1/Q)\varepsilon^{\alpha\beta}\varepsilon_{\rho\delta}$ and $\mathcal{P}_{\mathcal{S}\rho\delta}{}^{\alpha\beta} = \delta^\alpha_\rho\delta^\beta_\delta - \mathcal{P}_{\mathcal{A}\rho\delta}{}^{\alpha\beta}$ as $R^{\pm\alpha\beta}_{\rho\delta} = \mathcal{P}_{\mathcal{S}\rho\delta}{}^{\alpha\beta} - a^{\pm2}\mathcal{P}_{\mathcal{A}\rho\delta}{}^{\alpha\beta}$ we may decompose the $\mathcal{R}^{\pm NM}_{LK}$ as a sum of four projectors [9]

$$\mathcal{R}^{\pm NM}_{LK} = \mathcal{R}^{\pm NM}_{\mathcal{S}KL} - a^{\pm2}\mathcal{R}^{\pm NM}_{\mathcal{A}KL} - a^{\mp2}\mathcal{R}^{\pm NM}_{\bar{\mathcal{A}}Kl} + \mathcal{R}^{\pm NM}_{\mathcal{T}KL}, \quad (56)$$

where $\mathcal{R}^{\pm}_{\mathcal{T}}$ is the quantum trace projector, $\mathcal{R}^{\pm}_{\mathcal{S}}$ is the traceless part of the quantum symmetrizer, and $\mathcal{R}^{\pm}_{\mathcal{A}}$ and $\mathcal{R}^{\pm}_{\bar{\mathcal{A}}}$ are the selfdual and antiselfdual parts of the quantum antisymmetrizer given respectively by

$$\mathcal{R}^{\pm NM}_{\mathcal{S}LK} = \frac{1}{Q^2} R^{\mp\alpha\kappa}_{\tau\sigma} \mathcal{P}_{\mathcal{S}\kappa\nu}{}^{\mu\beta} \mathcal{P}_{\mathcal{S}\delta\alpha}{}^{\rho\varepsilon} R^{\pm\lambda\gamma}_{\varepsilon\mu} \sigma^{M}_{\gamma}{}^{\dot\beta} \sigma^{N}_{\rho}{}^{\dot\lambda} \bar\sigma_{L\dot\tau}{}^{\delta} \bar\sigma_{K\dot\nu}{}^{\sigma}, \tag{57}$$

$$\mathcal{R}^{\pm NM}_{\mathcal{T}LK} = \frac{1}{Q^2} R^{\mp\alpha\kappa}_{\tau\sigma} \mathcal{P}_{\mathcal{A}\kappa\nu}{}^{\mu\beta} \mathcal{P}_{\mathcal{A}\delta\alpha}{}^{\rho\varepsilon} R^{\pm\lambda\gamma}_{\varepsilon\mu} \sigma^{M}_{\gamma}{}^{\dot\beta} \sigma^{N}_{\rho}{}^{\dot\lambda} \bar\sigma_{L\dot\tau}{}^{\delta} \bar\sigma_{K\dot\nu}{}^{\sigma}, \tag{58}$$

$$\mathcal{R}^{\pm NM}_{\mathcal{A}LK} = \frac{1}{Q^2} R^{\mp\alpha\kappa}_{\tau\sigma} \mathcal{P}_{\mathcal{S}\kappa\nu}{}^{\mu\beta} \mathcal{P}_{\mathcal{A}\delta\alpha}{}^{\rho\varepsilon} R^{\pm\lambda\gamma}_{\varepsilon\mu} \sigma^{M}_{\gamma}{}^{\dot\beta} \sigma^{N}_{\rho}{}^{\dot\lambda} \bar\sigma_{L\dot\tau}{}^{\delta} \bar\sigma_{K\dot\nu}{}^{\sigma}, \tag{59}$$

$$\mathcal{R}^{\pm NM}_{\bar{\mathcal{A}}LK} = \frac{1}{Q^2} R^{\mp\alpha\kappa}_{\tau\sigma} \mathcal{P}_{\mathcal{A}\kappa\nu}{}^{\mu\beta} \mathcal{P}_{\mathcal{S}\delta\alpha}{}^{\rho\varepsilon} R^{\pm\lambda\gamma}_{\varepsilon\mu} \sigma^{M}_{\gamma}{}^{\dot\beta} \sigma^{N}_{\rho}{}^{\dot\lambda} \bar\sigma_{L\dot\tau}{}^{\delta} \bar\sigma_{K\dot\nu}{}^{\sigma} \tag{60}$$

and satisfy

$$\delta^N_L \delta^M_K = \mathcal{R}^{\pm NM}_{\mathcal{S}LK} + \mathcal{R}^{\pm NM}_{\mathcal{A}LK} + \mathcal{R}^{\pm NM}_{\bar{\mathcal{A}}LK} + \mathcal{R}^{\pm NM}_{\mathcal{T}LK}, \tag{61}$$

$$\mathcal{R}^{\pm NM}_{\mathcal{I}LK} \mathcal{R}^{\pm LK}_{\mathcal{J}PQ} = \delta_{\mathcal{I}\mathcal{J}} R^{\pm NM}_{\mathcal{I}PQ}, \quad \mathcal{I},\mathcal{J} = \mathcal{S},\mathcal{T},\mathcal{A},\bar{\mathcal{A}}. \tag{62}$$

From Eq. (49), the quantum symmetrization of the coordinates is given by

$$(\delta^N_L \delta^M_K - \mathcal{R}^{\pm NM}{}_{LK}) X_{\pm N} X_{\pm M} = 0 \;\Rightarrow\; \begin{cases} \mathcal{R}^{\pm NM}_{\mathcal{A}LK} X_{\pm N} X_{\pm M} = 0, \\ \mathcal{R}^{\pm NM}_{\mathcal{A}LK} X_{\pm N} X_{\pm M} = 0, \end{cases} \tag{63}$$

where we have used (56),(61) and (62). The $\mathcal{R}^{\pm}$ matrices satisfy the same Hecke condition (44) and the same decomposition in terms of projectors (56) as the $\mathcal{R}_{(I)}$ matrix of [9]. If we take a Minkowski space–time equipped with the coordinates $X_I = X_{+I}$ and a metric $G^{IJ} = G^{IJ}_{+}$ given in Section 3, the four-vector length is given by $-G^{IJ} X_I X_J = q^{-(1/3)} X_0^2 - q^{(1/2)} X_3^2 - (q^{(1/3)}/Q) Z\bar{Z} - (q^{-(1/2)}/Q) \bar{Z} Z$, where $Z = X_1 + \mathrm{i}X_2$ and $\bar{Z} = X_1 - \mathrm{i}X_2$. On the other hand, an explicit calculation gives from (63) the following commutation relations [14]

$$X_0 X_I = X_I X_0, \quad X_3 Z - q^2 Z X_3 = (q - q^{-1}) X_0 Z,$$
$$Z\bar{Z} - \bar{Z} Z = (q^2 - q^{-2}) X_3^2 + q^{-1}(q^2 - q^{-2}) X_0 X_3.$$

By redefining the Minkowski space–time coordinates as $C = qX_0 - X_3$, $D = q^{-1} X_0 + X_3$, $A = Z$ and $B = \bar{Z}$ we retrieve the commutation relations and the four-vector length given in [7,9,10,18].

Note that we can also have functionals $\mathcal{F}_{\pm L}{}^{M} = (1/Q)\, \tilde{f}_{\pm\dot\beta}{}^{\dot\alpha} \bar\sigma_{L\dot\alpha}{}^{\delta} \star f_{\pm\delta}{}^{\gamma} \sigma^{M}_{\gamma}{}^{\dot\beta} : \mathcal{A} \to C$ which controls the noncommutativity of the quantum Lorentz group. The action of $\mathcal{F}_{\pm L}{}^{K}$ on the generators $\Lambda_K{}^N$ gives the $\mathcal{R}^{\pm}_{(\mathrm{III})}$ matrices

$$\mathcal{R}^{\pm\, NM}_{(\mathrm{III})LK} = \frac{a^{\mp 2}}{Q^2} R^{\pm\alpha\kappa}_{\tau\sigma} R^{\pm\mu\beta}_{\kappa\nu} R^{\pm\rho\varepsilon}_{\delta\alpha} R^{\pm\lambda\gamma}_{\varepsilon\mu} \sigma^{M}{}_{\gamma}{}^{\dot\beta} \sigma^{N}{}_{\rho}{}^{\dot\lambda} \bar\sigma_{L\dot\tau}{}^{\delta} \bar\sigma_{K\dot\nu}{}^{\sigma}, \tag{64}$$

which satisfy Yang–Baxter equations but cannot be decomposed in terms of projecteurs (57)–(60). These functionals may be derived from right invariant base ${X'}_{\pm\alpha\dot\beta} = \theta_{\pm\alpha}\theta_{\pm\dot\beta}$ subject to the condition $({X'}_{\pm\alpha\dot\beta})^\star = {X'}_{\mp\beta\dot\alpha}$ leading to right invariant bases ${X'}_{\pm I}$ of an $\mathcal{L} - \mathcal{M}'_\pm$ bimodules satisfying $({X'}_{\pm I})^\star = {X'}_{\mp I}$. From these bases we may construct real coordinates of the Minkowski spaces $\tilde X_{\pm I} = {X'}_{\pm I} + {X'}_{\mp I}$ but the corresponding $\tilde R^\pm = \mathcal{R}^\pm_{\rm (III)} + \mathcal{R}^\mp_{\rm (III)}$ matrices do not satisfy the Yang–Baxter equations.

Another set of $\mathcal{R}$ matrices can be constructed out of the R matrix. In fact, from a purely algebraic point of view, we can start from the relation (52) where ${\Lambda_N}^M$ are of the form (29), then by using the commutation relations of the generators of $SL(2, C)$ we may show that the general solutions of the R matrix are given by

$$\mathcal{R}^{(abcd)NM}_{LK} = \frac{a^{-(a+b+c+d)/2}}{Q^2} R^{(a)\alpha\kappa}_{\tau\sigma} R^{(b)\mu\beta}_{\kappa\nu} R^{(c)\rho\varepsilon}_{\delta\alpha} R^{(d)\lambda\gamma}_{\varepsilon\mu} \sigma^{M\,\dot\beta}_{\gamma} \sigma^{N\,\dot\lambda}_{\rho} \bar\sigma^{\delta}_{L\dot\tau} \bar\sigma^{\sigma}_{K\dot\nu}, \tag{65}$$

where the indices $a, b, c, d = \pm$ or $\mp$. The Yang–Baxter conditions require $c = a$ or $c = d$ and $b = a$ or $b = d$. By using the completeness relations (27) we can show that the inverse is given by $\mathcal{R}^{-1(abcd)} = \mathcal{R}^{(-d-b-c-a)}$. A straightforward calculation gives

$$\left(\mathcal{R}^{(abcd)NM}_{LK}\right)^\star = \frac{a^{-(a+b+c+d)/2}}{Q^2} R^{(a)\alpha\kappa}_{\tau\sigma} R^{(c)\mu\beta}_{\kappa\nu} R^{(b)\rho\varepsilon}_{\delta\alpha} R^{(d)\lambda\gamma}_{\varepsilon\mu} \sigma^{N\,\dot\beta}_{\gamma} \sigma^{M\,\dot\lambda}_{\rho} {\bar\sigma_{K\dot\tau}}^{\delta} {\bar\sigma_{L\dot\nu}}^{\sigma}. \tag{66}$$

Then in addition to $\mathcal{R}_{\rm (III)} = \mathcal{R}^{++++}$, its inverse $\mathcal{R}^{-1}_{\rm (III)} = \mathcal{R}^{----}$ and $\mathcal{R}^\pm_{\rm (I)} = \mathcal{R}^{\mp\mp\pm\pm}$ and their inverse $\mathcal{R}^{-1\pm}_{\rm (I)} = \mathcal{R}^{\mp\pm\mp\pm}$ we also have solutions of the form $\mathcal{R}^{\mp\pm\pm\pm}$ and their inverse $\mathcal{R}^{\pm\pm\pm\mp}$. The condition (66) shows that among the different $\mathcal{R}$ matrices only $\mathcal{R}^\pm_{\rm I}$ satisfy the relation $(\mathcal{R}^{NM}_{LK})^\star = {\mathcal{R}^{-1}_{KL}}^{MN}$ required by (Theorem 3.2b) the consistency conditions between commutation relations (52)–(54) and the reality of the generators of the Lorentz group and the Minkowski space–time coordinates. The $\mathcal{R}_{\rm (III)}$ and $\mathcal{R}^\pm_{\rm (II)}$ matrices are consistent with the reality conditions of the commutation relations (52) and (54) but not with (53) because $({\mathcal{R}_{\rm (III)}}^{NM}_{LK})^\star = {\mathcal{R}_{\rm (III)}}^{MN}_{KL}$ and $(\mathcal{R}^{\pm\,NM}_{\rm (II)LK})^\star = \mathcal{R}^{\pm\,MN}_{\rm (II)KL}$ while the reality conditions of (53) require $(\mathcal{R}^{NM}_{LK})^\star = \mathcal{R}^{-1MN}_{KL}$. We can associate to the $\mathcal{R}^{\mp\pm\pm\pm}$ matrices the functionals $\mathcal{F}^{\pm\,M}_{\rm (II)L} : \mathcal{L} \to C$ such that $\mathcal{F}^{\pm\,M}_{\rm (II)L}({\Lambda_K}^N) = \mathcal{R}^{\pm\,NM}_{\rm (II)LK} = a^\mp \mathcal{R}^{\mp\pm\pm\pm}$. The $\mathcal{R}^\pm_{\rm (II)}$ matrices may be decomposed in terms of projectors (57)–(60)

$$\mathcal{R}^{\pm\,NM}_{\rm (II)LK} = a^{\mp 2}\mathcal{R}^{\pm NM}_{SLK} + a^{\pm 2}\mathcal{R}^{\pm NM}_{\mathcal{T}LK} - \mathcal{R}^{\pm NM}_{\mathcal{A}LK} - \mathcal{R}^{\pm NM}_{\bar{\mathcal{A}}LK} \tag{67}$$

and satisfy the Hecke relations

$$(\mathcal{R}^\pm_{\rm (II)} + 1)(\mathcal{R}^\pm_{\rm (II)} - a^{\pm 2})(\mathcal{R}^\pm_{\rm (II)} - a^{\mp 2}) = 0 \tag{68}$$

as the $\mathcal{R}_{\rm (II)}$ matrix of [9]. But in this case, we cannot write the functionals $\mathcal{F}_{\rm (II)}$ in terms of ${f_{\pm\alpha}}^\beta$ and ${\tilde f_{\pm\dot\alpha}}^{\dot\beta}$ and therefore the right invariant bases $X''_{\pm I}$ of the $\mathcal{L} - \mathcal{M}''_\pm$ bimodules cannot be written as a bilinear of spinors and conjugate spinors.

Acknowledgements

I am grateful to M. Dubois-Violette for his kind interest and helpful suggestion. I also thank N. Touhami for valuable discussions and S. Balaska for his help which allowed me to have results from the computer MAPLE program.

References

[1] A. Connes, Publ. IHES 62 (1986) 257.
[2] R. Coquereaux, Noncommutative geometry and theoretical physics, J. Geom. Phys. 6 (1989) 425–490.
[3] M. Dubois-Violette, R. Kerner, J. Madore, Noncommutative differential geometry of matrix algebras, J. Math. Phys. 31 (1990) 316–322.
[4] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups, Commun. Math. Phys. 122 (1989) 125–170.
[5] P. Podels, S.L. Woronowicz, Quantum deformation of Lorentz group, Commun. Math. Phys. 130 (1990) 381–431.
[6] P. Podles, S.L. Woronowicz, On the classification of quantum Poincaré group, Commun. Math. Phys. 178 (1996) 61–82.
[7] U. Carow-Watamura, M. Schlieker, M. Scholl, S. Watamura, Tensor representation of the quantum group $SL_q(2, C)$ and quantum Minkowski space, Z. Phys. C 48 (1990) 159.
[8] W.B. Schmidke, J. Wess, B. Zumino, A q-deformed Lorentz algebra, Z. Phys. C 57 (1991) 495–518.
[9] O. Ogiesvestsky, W.B. Schmidke, J. Wess, B. Zumino, q-Deformed Poincaré algebra, Commun. Math. Phys. 150 (1992) 495–518.
[10] M. Pillin, L. Weikl, On representation of the q-deformed Lorentz and Poincaré algebras, J. Phys. A 27 (1994) 5525–5540.
[11] M. Dubois-Violette, G. Launer, The quantum group of a non-degenerate bilinear form, Phys. Lett. B 245 (1990) 175–178.
[12] M. Lagraa, Quantum gauge theories, Int. J. Mod. Phys. A 11 (1996) 699–713.
[13] A.B. Hammou, M. Lagraa, The B.R.S.T. operator of quantum symmetries, the quantum analog of the Donaldson invariant, J. Math. Phys. 38 (1997) 4462–4472.
[14] M. Lagraa, On the measurability of observables in noncommutative special relativity, Math.ph/9904014.
[15] A. Connes, Noncommutative Geometry, Academic Press, New York, 1994.
[16] R. Coquereaux, G. Esposito-Farese, G. Vaillant, Higgs fields as Yang–Mills fields and discrete symmetries, Nucl. Phys. B 353 (1991) 689–706.
[17] M. Dubois-Violette, R. Kerner, J. Madore, Noncommutative differential geometry and new models of gauge theory, J. Math. Phys. 31 (1990) 323–330.
[18] U. Carow-Watamura, M. Schlieker, M. Scholl, S. Watamura, A quantum Lorentz group, Int. J. Mod. Phys. A 6 (1991) 3081–3108.
[19] H.B. Benaoum, M. Lagraa, $U_q(2)$ Yang–Mills theory, Int. J. Mod. Phys. A 13 (1998) 553–568.